Dynamic Cardinality Constrained Portfolio Optimization with fixed and linear Transaction Costs

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I would like to whole-heatedly dedicate this paper to my parents, family and friends[1] for their support, listening, feedback and encouragement and all the other people who have helped me in one way or another to complete this dissertation. Without them, I could hardly make this project done.

[1] Including but not limited to Miss Sirius Wang, Miss Can Huang, Miss Qinyuan Huang, Miss Cici Mao, ...

Declaration

I hereby declare that except where specific reference is made to the work of others, the thesis "Dynamic Cardinality Constrained Portfolio Optimization with fixed and linear Transaction Costs" submitted for the degree of Bachelor in Engineering to the Chinese University of Hong Kong has not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other University. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except where specifically indicated in the text.

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Abstract

A direct application of traditional Markowitz portfolio selection theory is problematic, especially for individual investors with small amount of initial capital available because of the unrealistic assumptions in the classical model. Existence of transaction costs in the form of bank and broker fees in reality forces investors to include only a rather small selection of assets in their portfolios which is in the form of **cardinality constraint portfolio optimization problem**.

Given the facts that previous efforts paid mainly on putting side constraints such as turnover constraints or number of trades limits constraints, few of the existing methods could guarantee an optimal solution to a dynamic Cardinality Constrained Portfolio Optimization Problem with Transaction Costs. In addition, since static general CCMV problems are already solved for a single time period in my previous work, I was motivated to investigate on **re-balancing (dynamic)** CCMV problems incorporated with transaction costs problem particularly in this semester.

In this thesis, I present the formulation based on the Markowitz MV model for rebalancing an existing portfolio subject to both cardinality and transaction costs constraints. I determine and demonstrate modellings of both fixed and linear transaction cost types which are the most common practices in the real world. I compare and contrast the effects of various constraints on the portfolio performance and derive the underlying implications and deep principles not only for such Dynamic Cardinality Constrained Portfolio Optimization with fixed and linear Transaction Costs models, but also applicable for the majority of portfolio optimization problems in general.

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List of Symbols

This list gives a description of the symbols and notations used throughout the dissertation.

Parameters

W_0	Investor's initial wealth, i.e, initial investment capital to start the investment
N	Total number of risky assets available in the investment universe
f	the desired portfolio rebalancing frequency, could be daily, monthly, quarterly or yearly
K	the desired number of distinct risky assets the investor wish to hold in the portfolio over time periods T
Δ	the desired number of changes in position in the new portfolio, could be even numbers only, ranging from 0 to 2K

Input

$\mu_{t,i}$	the expected return vector of the n risky assets, i.e, the expected return of per unit capital invested on asset i at period t, estimated by historical samples
$\sigma_{t,ij}$	the covariance matrix between the return of asset i and asset j (i, j = 1,. . . ,N) at time point t
$c_{t,i}$	the transaction cost rate paid if we perform any trades of asset i (i = 1,. . . ,N)
$fc_{t,i}$	fixed transaction cost rate on asset i at time point t
$l_{t,i}$	lower bound on portfolio weight of asset i in the new portfolio at time point t
$lc_{t,i}$	linear transaction cost rate on asset i at time point t
$P_{t,i}$	the current market price per share of asset i

$u_{t,i}$ upper bound on portfolio weight of asset i in the new portfolio at time point t

$X_{t,i}$ the number of shares hold in the current portfolio of asset i (i = 1,. . . ,N) at time point t

Intermediary results

$A_{t,i}$ binary variable representing whether asset i is included in the current portfolio or not at time point t, equals to 1 if it is, 0 otherwise

V_t monetary value of the current portfolio

v_t monetary value of the new portfolio

TC_t the total transaction costs incurred in order to move from the current portfolio to the new optimal one

Decision variables

$a_{t,i}$ binary variable representing whether asset i is to be included in the new portfolio at time point t, equals to 1 if it is, 0 otherwise

$t_{t,i}$ the number of shares to trade in the position on asset i in order to get the optimal portfolio at time point t

$w_{t,i}$ portfolio weight vector of asset i in the new portfolio at time point t

$x_{t,i}$ the number of shares to hold in the portfolio of asset i in the new portfolio at time point t

Chapter 1

Introduction

1.1 Thesis Structure

The paper is organized as follows. In this chapter, I first introduce the thesis structure and the problem of interest of this project, followed by review of works that have been done in the previous semester and then point out the road-map for new semester.

In Chapter 2, I present a literature review of the history of portfolio optimization practices with particular focus on works involving cardinality constraints and transaction costs constraints. Moreover, the review is dynamic-rebalancing focused in the aim of reviewing how scholars have been tackling with portfolio re-balancing problems.

Chapter 3 is the detailed model formulation and is the main body of this thesis . Specifically, I first explain the model setup in detail and then build up the objective function. Detailed explanations of each of the constraint that appears in the model are given in the next. After that, I point out several straightforward facts underlying the model structure and analyse relaxation conditions to release computational burden and improve practicability. In the last part of this chapter, I give completes formulation of the dynamic cardinality constrained portfolio optimization model with fixed and linear transaction costs.

I present the results of applications of my models on the chosen data sets in Chapter 4. In detail, various portfolios are compared with each other in terms of both static and dynamic performances. I present graphical illustrations of the efficient frontiers for the static performance and rolling monetary value of the portfolio for dynamic performance. After these, various investment strategies are computed to carry out the sensitivity analyses on different parameter settings and to study effects of one or some of the constraints.

Chapter 5 is a conclusion of the thesis presenting a summary of the thesis, advances of the model, potential deficiency and future directions. In addition, I explained other applications of the model and my contribution to knowledge.

1.2 Problem Review

In this final year project, I have been particularly interested in solving the Cardinality Constrained Portfolio Selection problem. Based on Markowitz's Mean Variance Portfolio Optimization framework, investors would always allocate in as many as risky assets available in the market to fully diversify away risks. This solution to Markowitz's Mean Variance Model (MV model), however, is usually unachievable in real world due to various unrealistic assumptions.

First, Mean Variance model assumes an ideal market with no frictions, such as transaction costs or management fees. This, however, is more or less violated in the real world. In the past decades, brokers make money on commissions and fees no matter the gains or losses sustained by individual traders. Such transaction costs or management fees, which are out of consideration in MV model are exactly a serious issue in real life. A direct application of the classical portfolio selection theory is more problematic for small investors as the large amount of transaction fees would eat up some or even all of their capitals. [1] They are hence in favorable to include a comparatively rather small number of assets in their portfolio. We give a graphic proof of this statement in Chapter 4.

Second, shorting is not allowed in this original model, allocation on each asset could be non-negative only. [14] Situation is much more complex in real world. Despite the heavy scrutiny felled on short selling in many countries, especially after the global financial crisis in 2007-2008, shorting is still allowed in many countries, usually with a higher short selling cost however.

As a result, such unrealistic limitations on the original model have motivated a number of scholars to investigate on **cardinality constrained mean-variance portfolio selection (CCMV)** problem. The results of studies on the diversification effect of portfolio size[1] further strengthen the importance of CCMV problem: decrease in portfolio risk due to the increase of portfolio size would diminishing once it has achieved a certain level, according to Elton and Gruber's research paper. In one word, it is both important and meaningful to solve the **cardinality constrained mean-variance portfolio selection (CCMV)** problem.

1.3 Previous Achievements

In the previous semester, after a wide range review of previous studies on cardinality constrained portfolio optimisation problem, I was dedicated to develop a scientific-based heuristic algorithm that integrates factor models in finance, clustering analysis in computer science

[1] See section 3.2.3 for more detailed elaboration of the study results.

and mixed integer programming models in operations research to solve CCMV problem based on Jiang's [4] proposal. To this end, various market analyses were carried out to find the most appropriate number of clusters for different markets. In addition, particular attention was paid on establishing a muti-factoe regression model by analysing the potential various factors that are highly correlated to asset returns. I then group highly correlated risky assets based on their factor loadings in linear regression model and solve the CCMV problem by heuristics.

In more detail, I first evaluated Jiang's algorithm in achieving feasibility and optimality by implementing the factor model suggested in the paper using the same data set in Hong Kong market and concluded the conditions where such heuristics would deliver correct solution as the direct methods within certain confidence level. I then build up original models with various combinations of analyses on factors such as industrial factors, extended Fama-French 4 factors and Buffet's six factors with three clustering algorithms. Such models have never appeared in former literature and is a complete original work. After that, further assessments of the models were conducted by out-of-sample backtests in comparison with the benchmark strategy.

1.4 Road-map for new efforts

During the process of deeper and further investigation of the problem, I realize the genuine need for constructing a cardinality constraint for portfolio optimization problem lies in the existence of transaction fees. As a result, I was passionate about injecting the transaction costs into the original CCMV model.

With the former effort in solving static CCMV solutions, the model was able to give a static solution given **independent** data input. However, the optimization problem in real life is much more complex in the sense that investors usually hold a portfolio for a long period of investment horizon, frequently check their portfolio and adjust accordingly. Thus, a dynamic CCMV re-balancing model is more applicable for the reality.

Mathematically, this is a problem involving much more complexity. Now, each optimal solution becomes a random variable and is **dependent** on the existing, or current, portfolio. The stock positions in current portfolio has at least some decision power, if not all , on the optimal portfolio to be hold in the next time period, especially for the frictional market. Intuitively, if the CCMV model gives a mean-variance optimal solution with totally different stocks positions, namely, clear all of the current positions and open all new positions, which means 200% turnover for the entire portfolio. The cost related to this trade activity is tremendously high. Hence we have to balance the costs and benefits in rebalancing portfolio: to decide whether, when and how to rebalance.

To these ends, I have planned to do and indeed have completed the following:

- Develop and apply a more generalized and sensible factor model in the sense that the specific returns are more white and could better predict the return.
- Modify sample covariance measure into Ledoit-wolf shrinkage covariance measure to derive a more stable and computable covariance matrix.
- Establish a dynamic portfolio optimization model for rebalancing management. [2]
- Injecting transaction costs into portfolio optimization model to consider a net in transaction cost portfolio return. [3]
- Combine and test different optimization models and evaluate their performances by putting more realistic objective functions and bounding constraints to improve both the efficiency, feasibility and profitability of the portfolio optimization. [4]

[2]See Chapter 3
[3]See Chapter 3
[4]See Chapter 4

Chapter 2

Literature Review

In this Chapter, we would give a review of previous studies on portfolio re-balancing problems with cardinality constraints and transaction cost constraints.

2.1 Cardinality Constrained Portfolio Optimisation

Work on cardinality constraints mean variance model in the last two decades can be generally classified into two categories, namely, direct and heuristic algorithms.

Direct methods refer to scholars who directly tackle the problem, provide possible solutions from mainly pure mathematical or theoretical angle. Heuristic methods, on the other hand, rely on "domain knowledge from a particular application, that gives guidance in the solution of a problem" (Oxford Dictionary of Computing, 1996). Please refer to Reeves [9] for more information about **heuristic**.

While Li, Sun, and Wang (2006) [5] and a few others assumed a concave transaction costs function to find an exact solution for cardinality-constrained mean-variance problem by applying convergent Lagrangian methods assuming concave transaction costs function, the majority of the research community focusing on **heuristic algorithms**. For example, Crama and Schyns (2003)[2] implemented simulated annealing Algorithms in the aim of solving complex portfolio selection models with cardinality constraints. To get feasible solutions, they also put side constraints such as turnover constraints to limit the number of trades. Jiang et al (2014) [4] applied a scientific-based heuristic algorithm which integrates factor models in finance, clustering analysis in computer science and mixed integer programming models in operations research to solve CCMV problem. Specific interests were put on this smart proposal in the last semester since it offered such a unique and intelligent approach to solving CCMV problem by eliminating the original cardinality constraint in their mortified model. Diligent examination on this approach is given in the previous section (See section

1.3) Soleimani et al (2009) [11] grouped various assets into different sectors and established a sector-grouped model with weight constraints in each sector to solve the cardinality constrained problem by genetic algorithm.

2.2 Portfolio optimization with Transaction Cost

Transaction costs, simply put, are additional expenses incurred in order to make the transaction done. In financial world, transaction costs usually include brokers' commissions, various fees and spreads [1]. While these transaction costs are major profits banks and brokers receive for their roles, they are a considerable amount of payments for investors and should never be sniffed at. Transaction costs are so important because they are one of the key determinants of net returns, which also explains the large gap between net return in real life against the theoretical frictionless return in Mean Variance model. Transaction costs diminish returns, and over time, high transaction costs can mean thousands of dollars lost from not just the costs themselves but also because the costs reduce the capital available for future investment.[8]

In the academic arena, usually two types of transaction cost function have been considered: fixed and linear transaction cost. Satchell [10] gave three common alternative transaction cost models which are shown figure 2.1 in including a linear cost model with a fixed marginal rate (Model 1 in the figure), a linear cost model with a fixed hurdle rate for making the purchase (Model 2) and models in higher orders (Model 3 in the figure is a quadratic one).

While the globally optimal portfolio can be computed rather rapidly for linear transaction costs, it is a different story for the fixed costs. Miguel Sousa Lobo et al (2007) [7] considered portfolio selection problem, with transaction costs constraints and constraints on exposure to risk. They developed a relaxation method which computes an upper bound via convex optimization and derived suboptimal solutions for fixed transaction costs function model. Baule [1] assumes a non-convex transaction cost function and considers a model with a trade-off between transaction costs and risk costs for different levels of invested wealth. They analyze the reformulated optimization problem with an objective function as a minimization of the sum of two types of costs and conclude that transaction costs can lead to a rather small optimum for the number of stocks in the portfolio, especially for small investors.

[1] The differences between the price the dealer paid for a security and the price the buyer pays.

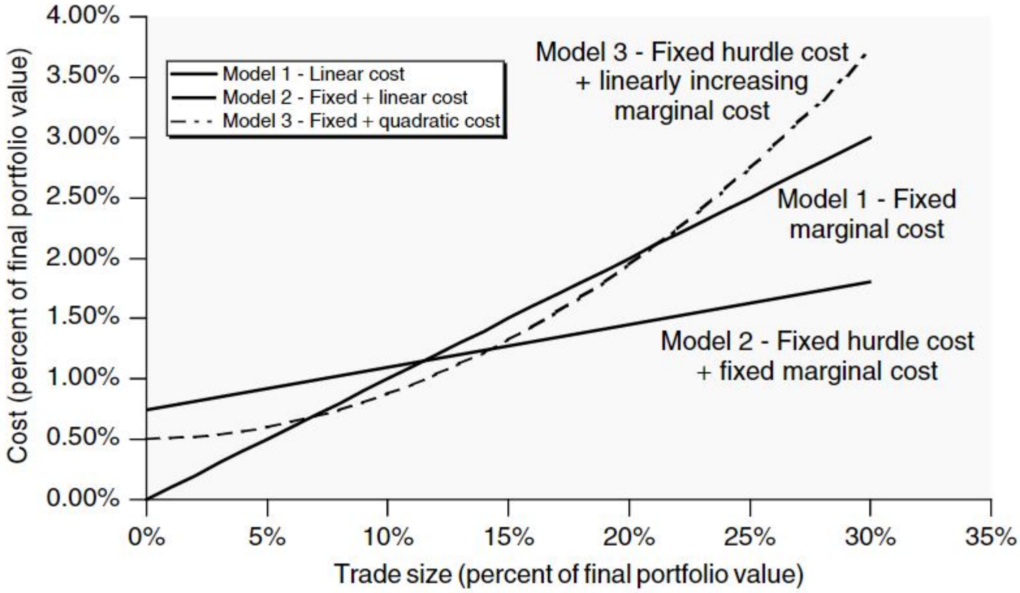


Fig. 2.1 Common Transaction Cost Models

2.3 Summary

This chapter gave a glimpse of previous researches on portfolio optimization problem with particular attention on the cardinality constraint, transaction cost constraint with a dynamic re-balancing viewpoint. As a result of the literature review above, there are several findings and considerations.

I noticed that several ways, either by clustering algorithm or injecting appropriate transaction cost constraints could eliminate the cardinality constraints in CCMV problem, to achieve the same desired solution with only a small number of stocks in the optimal portfolio.

There are a bunch of benefits of these side constraints. To name a few, it may help solve the CCMV, which is NP-hard, in a more fast and efficient way. Also, it increases computational feasibility and facilitates efficiency, saves computational time and thus increasing solution speed. This is especially important for financial industry, where the saying of "time is money" cannot be more true. However, these benefits do come at costs. The biggest problem is that the cardinality number K is not fixed. Although side constraints such as turnover constraints could limit the portfolio size to be a small number, but that is not stable but changes from time to time. For example, consider a portfolio without cardinality constraint but only with transaction cost constraints, there may be 5 stocks in the portfolio at the current time point, but 6 in the previous, and may be 4 in the coming investment period, which is still a large and important gap between the original CCMV solution.

Chapter 3

Model Formulation

Reflection on results of work in the last semester guided me to conclude that if we are able to build up a better formulated optimization problem model to the degree where the auxiliary multi-factor model does not change the solution a lot then the problem would be solved in a more artistic, concise and efficient way.

This insight motivates me to investigate on re-balancing (dynamic) CCMV problems incorporate with transaction costs. This is different compared with the focus in the previous work. To elaborate , recall that full attention was focused on finding the static cardinality solution within each time period. Either conducting market analysis to find the most appropriate number of clusters for different markets or establishing various factors, building up multi-factor models and then grouping highly correlated risky assets based on their factor loadings in linear regression model could solves the CCMV problem once only. In other words, any time investors want to rebalance their portfolios, the procedures mentioned above have to repeat again and again, which is obviously inefficient.

However, it would be another story if the new re-balancing (dynamic) CCMV problems incorporate with transaction costs could be successfully modelled. Solutions would be updated automatically suggesting investors whether, when and how to trade by considering the costs and benefits of the potential transaction behaviours. In brief, it was a static solution in the last semester while a dynamic and updated model in this paper.

Given that there exists no exact re-balancing (dynamic) solutions to CCMV with transaction costs problems according to the literature review, the whole modified CCMV net in transaction costs effect models that are going to be illustrated in this chapter is originally developed and totally brand-new.

3.1 Model Setup and objective function

Refer to the list of Symbols at the beginning of the paper for a complete collection of symbols and notations that used throughout the thesis.

Suppose that an investor with initial wealth W_0 selects K risky securities from an investment universe consisting of N risky assets for constructing a portfolio at time point $T = 0$ and hold it over a fixed time period. The investor then choose to reallocate his or her posterior wealth at the beginning of each of the following $T - 1$ re-balancing time points by performing a number of transactions to adjust the weight proportions invested in each risky assets while satisfying a set of constraints on the portfolio which are going to be explained in details in the following sections. The investor's goal is to minimize portfolio risk in terms of "mean-variance" framework within each time period.

More precisely, the objective function is

$$\underset{a_{t,i},x_{t,i},w_{t,i},t_{t,i}}{\text{minimise}} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_{t,ij} w_{t,i} w_{t,j}$$

On one hand, the quadratic risk objective function is in consistent with the original Markowitz model in the sense of minimizing the summation of the covariance matrix entries multiplied by the weight proportion to be allocated in each risky asset i (i = 1,...,N). On the other hand, there are tiny yet important differences. Notice here in the set up the posterior wealth v_t is updated by subtracting the total transaction costs incurred in the process of moving from current portfolio holdings X_i to the new portfolio solution x_i and that is different from the anterior wealth V_t. The detailed mathematical equation for weight vectors are given in the sections below.

3.2 Constraints

In this section, I give detailed explanations of each constraint imposed on the portfolio. Since logic and procedure are the same for each rebalancing period, I would focus on a single time portfolio selection problem at time t in this explanation and then back to the dynamic model in full model formulation section.

3.2.1 Targeted return constraints

$$\sum_{i=1}^{N} \mu_{t,i} w_{t,i} = R \tag{3.1}$$

The return equation says that the expected portfolio return is a weighted sum of expected returns of assets currently hold in the portfolio since weight vector has non-zero entries for assets are currently in the portfolio only and zero otherwise. See equation 3.16.

3.2.2 Balance on shares

$$x_{t,i} = X_{t,i} + t_{t,i}, \quad i = 1, \ldots, N \tag{3.2}$$

The number of shares to hold on asset i in the new portfolio should equal to the sum of the current holdings of the same risky asset plus the corresponding trading amount of shares, where one of the decision variables $t_{t,i}$ denotes the trading vector or transaction amounts. Positive $t_{t,i}$ represents the units to buy while negative denotes the amounts to sell.

3.2.3 Cardinality constraints

$$\sum_{i=1}^{N} a_{t,i} = K \tag{3.3}$$

This equation limits the total number of positions open in the portfolio to be integer K, which is given by the investor.

Elton and Gruber [3] had published some interesting findings on the number of portfolio size, which is briefly explained in the following. Figure 3.1 highlights the relationship between number of securities holdings in the portfolio and portfolio total risk. It is clear to observe that the portfolio risk goes down as the portfolio size increases, however, the benefits of diversification diminishes. Noticing that total risk of the portfolio in the last column never goes to zero but converges to a stable level of at around 7.100, which represents the non-systematic risk that can not be diversified anyway.

The effect of diversification, namely, holding more types of uncorrelated risky assets, is better presented in chart 3.2: when there are only one asset in the portfolio, holding one more could reduce nearly half of the original risk, three more to reduce around two-thirds, so and so force, but the benefits of holding a bigger portfolio decline rapidly after n=30.

Thus, the choice of an appropriate K for the number of risky assets in the portfolio is a combination of art and science. For one thing, investors could choose to trust the market and and pick the number according to the number of low-correlated assets groups based on clustering analyses as what I had done in the last semester. For another, since fewer asset classes in a portfolio implying more expected alpha and more idiosyncratic risk whereas a larger portfolio size generally lowers the portfolio risk and return simultaneously through diversification , it largely depends on the investor's risk attitudes.

Table 8
Effect of Diversification

Number of Securities	Expected Portfolio Variance	Variance in Variance	Total Risk
1..........	46.619	1,411.041	46.811
2..........	26.839	201.963	26.934
4..........	16.948	31.553	16.996
6..........	13.651	11.184	13.683
8..........	12.003	5.477	12.027
10..........	11.014	3.186	11.033
20..........	9.036	.623	9.045
50..........	7.849	.075	7.853
100..........	7.453	.013	7.455
200..........	7.255	.001	7.256
500..........	7.137	.000	7.137
1,000..........	7.097	.000	7.097
Minimum......	7.070	.000	7.070

NOTE.—Parameters based on 3,290 securities values shown in table 5.

Fig. 3.1 Effect of Diversification

3.2.4 Position change constraints

This model grants even more freedom for investors to build their very customized portfolio as they are able to further specify the number of changes in positions they want for the portfolio and this is formulated as:

$$\sum_{i=1}^{N} |a_{t,i} - A_{t,i}| = \Delta \tag{3.4}$$

where

$$\Delta = 0, 2, 4, \ldots, 2K \tag{3.5}$$

The desired level of position changes Δ is an integer explicitly pointed out by the investor prior to the whole investment horizon. Due to its practical context, Δ could only be even numbers ranging from 0 to twice of the total number of risky assets included in the portfolio with the assumption that the desired number of assets hold in the portfolio K is much smaller than $N/2$. Δ could not be odd integers in the aim of maintaining the cardinality constraint simultaneously. If one decides to close one of he position in the current portfolio, he must also open a new one from those which are not in the portfolio , in other words, the portfolio size changes.

Here is an illustration example for better understanding. Consider a simple portfolio with risky assets universe $N = 6$ including AAPL, BA, GM, IBM, DD and GOOG. The desired portfolio size chosen by an individual investor is $K = 4$. Figure 3.3 gives a comparison

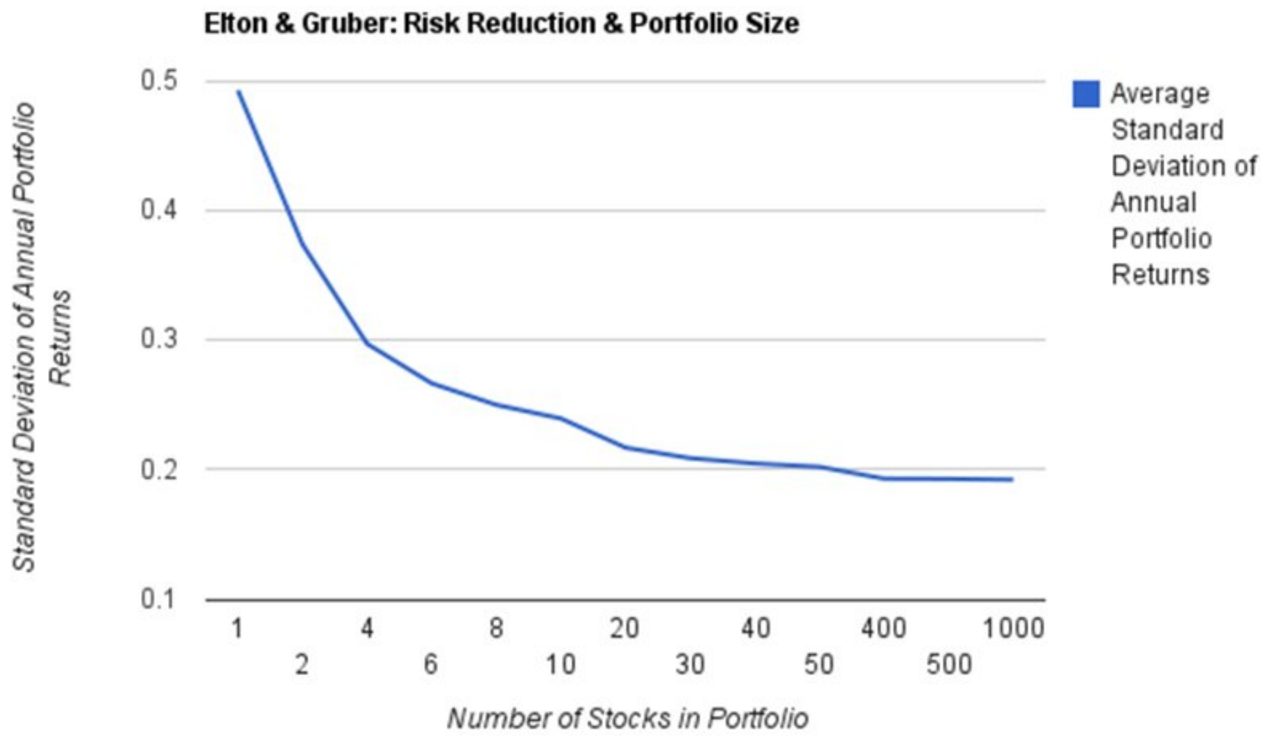


Fig. 3.2 Risk reduction and portfolio size

between current and new portfolio weight vectors at a specific rebalancing time point t. These numbers are randomly computed only for illustration purpose but entries of these two weight vectors sum up to one, satisfying the *fully invested constraint*. Observing the model suggests the investor to open a new position for AAPL and sell out all of his or her GOOG holdings to close the position, while maintaining positions on all of the other risky assets. All of the other stocks would involve non-significant transactions: either buying more units of current holding positions, selling less than current holdings shares or simply keeping unchanged. The key point here, however, is that there are only 2 position changes, namely, open AAPL and close GM. This piece of information is of particular importance for computing fixed transaction costs which would be fully explained later. To this end, we introduce two binary vectors $a_{t,i}$ and $A_{t,i}$ to denote whether asset i is included in the current portfolio or not and whether it is to be included in the new portfolio at time point t or not respectively. Hence, the situation in this example could be summarized by the current and new position index given in table 3.1. Noticing that the sum of the two rows equal to 4, which satisfies the cardinality constraint and LHS of the position change constraint (see equation 3.4) equals to 2. [1]

[1] Taking the second row of “new position index" minus that of the “current position index", take absolute value of each entry and then take the summation, you will get the same answer.

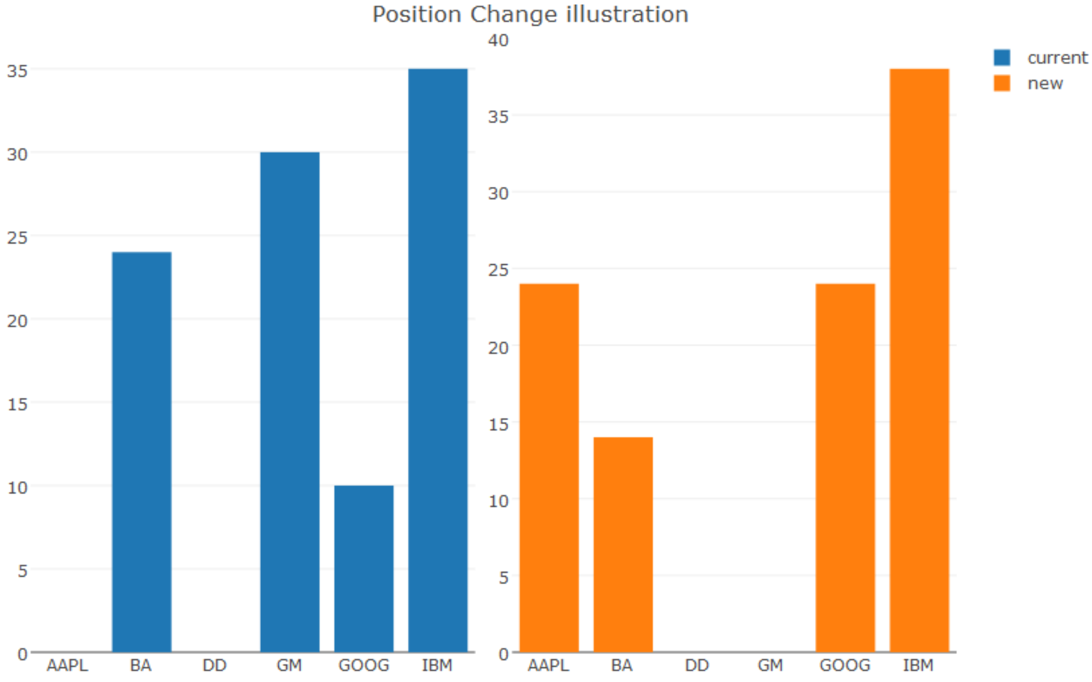


Fig. 3.3 Illustration Example: Current and new holding positions in a simple portfolio

Table 3.1 Example to illustrate how position change constraint works

position index	AAPL	BA	GM	IBM	DD	GOOG
current position index	0	1	0	1	1	1
new position index	1	1	0	0	1	1

Up to this point, the logic behind this position change constraint is rather clear and range of possible values of Δ is also self-explainable. More specifically, there are two extreme cases along the spectrum. $\Delta = 0$ means that the investor wants no position changes from the current portfolio to the new one , in other words, he or she would neither open new positions nor close any but only fine-tuning the current holdings in the portfolio in the sense of either buying more shares of stocks which are already in the portfolio or sell some but not all of the shares for some asset i. While $\Delta = 2K$ represents the most rapid change of portfolio positions: the investor would sell out all of his or her current holdings of all the stocks now in the portfolio and considers to buy only new ones that are not in the portfolio, which implies a rather large amount of transaction costs.

Different levels of Δ appeal to different investors with different background. For example, it is reasonable to expect an **individual** investor with relatively small amount of capital to choose a Δ value closer to zero since the unnecessary yet large amount transaction fees would eat up a lot, if not all, of his or her capital . Whereas institutional investors or individuals with large amount of capital available may be indifferent for various Δ values as the reduction of transaction costs on the total expected return is little compared with the denominator that they are approximately negligible.

It may be easier to understand by imagining two sets to represent the whole risky assets universe. Let

$$In = \{\, i \mid \text{risky asset i is currently in the portfolio} \} \tag{3.6}$$

and

$$Out = \{\, i \mid \text{risky asset i is currently not in the portfolio but is included in the asset universe} \} \tag{3.7}$$

Then, for this example,

$$In = \{\text{BA, GM, GOOG, IBM}\}$$

$$Out = \{\text{AAPL, DD}\}$$

and the binary variable could be also writtern as:

$$A_{t,i} = \begin{cases} 1, & \text{if } i \text{ in } In \\ 0, & \text{e} \end{cases} \tag{3.8}$$

3.2.5 Balance on portfolio value

$$v_t = V_t - TC_t \tag{3.9}$$

where

$$V_t = \sum_{i=1}^{N} P_{t,i} X_{t,i} \tag{3.10}$$

Figure 3.4 helps to understand the equations above.

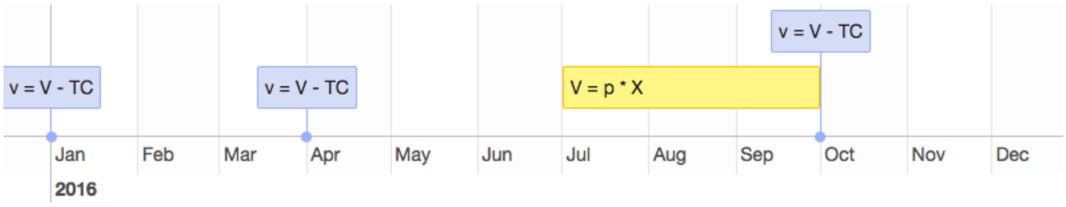


Fig. 3.4 Evolution of a quarterly rebalanced portfolio value

Equation 3.10 simply denotes that the portfolio value during any two rebalancing time points equals to the summation of monetary values of all the risky assets positions, which are multiplications of the market price of asset i and the number of shares of that asset hold in the portfolio. This is highlighted in the yellow box in figure 3.4. Whereas equation 3.9 states that the total value of the portfolio is updated by subtracting the incurred transaction costs moving from the current to the new portfolio which is expressed by the purple time points in figure 3.4. Noticing the portfolio value V_t is maintained for over time intervals while the updating of transaction costs is instantaneous which happens only at the beginning of each rebalancing time period.

Now, the task remains to model the transaction cost functions.

3.2.6 Transaction Costs modelling

Theoretically, both linear and non-linear transaction cost functions with any type of non-integer exponents coefficient could be computed and be incorporated into the model. However, considering that increasing complexity of transaction cost functions would cause a lot of drawbacks such as deepening computational burden, largely increasing problem solving time and making it unnecessarily hard for general investors to understand the model but bring about barely no sounding benefits except for uncertain accuracy, we decide to include linear transaction costs and only fixed transaction fees for non-linear transaction cost function in this paper. In addition, exact modelling of such functions is out of the scope of this paper and is left for those dear data scientists fellows.

Fixed transaction costs could be modeled as:

$$\sum_{i=1}^{N} fc_{t,i} |a_{t,i} - A_{t,i}| = TC_t \tag{3.11}$$

The total transaction costs are a summation of transaction costs on each risky asset position. Moreover, equation 3.11 states that transaction fees are incurred if and only if position changes take place. In other words, fine - tuning shares of the existing positions incurs no transaction costs and only opening new positions or closing old position would. Continuing the example given in the previous section, only the transactions on AAPL and IBM positions cause fee loses for the investor. If AAPL and IBM share a common fixed transaction cost rate of $1 per trade, then moving from the current portfolio to the new one makes $2 dollars losses in portfolio value in total for this investor.

Yet, the more popular practice of transaction costs implemented in world's most financial markets is linear. Linear transaction costs have numerous advantages for both practitioners and research communities. First, it is easy to understand, easy to write down and easy to implement. The most common cases of linear transaction costs are linear either on the number of shares or on the transaction dollar amounts. For example, TradeKing[2][12], one of the leading online brokers based in America charges $4.95 per trade for stocks whereas Hong Kong Exchange asks for "a Trading Fee of 0.005% per side of the consideration of a transaction (rounded to the nearest cent)". [13] For another, solutions to linear transaction costs are feasible and achievable for most computational engines to carry out. Here, in this model we choose the more complicated linear transaction cost function: linear on transaction amount to inject into the model and thus problems with linear on shares are self-evident.

Linear transaction costs are formulated as:

$$\sum_{i=1}^{N} lc_{t,i} P_{t,i} t_{t,i} = TC_t \tag{3.12}$$

where $lc_{t,i}$ is the linear transaction cost rate on asset i at time point t, usually in forms of basis point; $P_{t,i}$ is the market price vector for all risky assets at time point t and $t_{t,i}$ denotes the number of shares to trade on position i to achieve the new optimal portfolio.

Note however that there could exist different buy or sell cost rates. Not only various assets could have specific buy or sell cost rates that differ from each other but the rates may also vary from time to time. This assumption holds water in the real world as stocks in different industries are imposed of different regulations and commission fees charged by different financial intermediaries such as banks or index funds vary from each other.

Now with equation 3.9, 3.11 and 3.12, portfolio value is successfully updated in accordance with the transaction cost type and transaction details at each rebalancing decision time points, we could then compute the new portfolio weights vector accordingly.

[2] Access the website on Saturday 6th May, 2017, https://www.tradeking.com/rates

3.2.7 Constraints on weights

$$w_{t,i} = \frac{P_{t,i}x_{t,i}}{v_t}, \quad i = 1,\ldots,N \tag{3.13}$$

Weight proportion of asset i $w_{t,i}$ in the new portfolio is computed as the monetary value of position i, which is a multiplication of the market price $P_{t,i}$ and shares holdings $x_{t,i}$ in the new portfolio divided by the total portfolio new value.

In addition, we impose fully invested constraint on the weight vector:

$$\sum_{i=1}^{N} w_{t,i} = 1, \tag{3.14}$$

The sum of the proportional weight on each risky asset i must equal to one, which means that throughout portfolio investment time horizon, the investor is expected to invest all his or her money on the risky assets and would never withdraw any capital out from the portfolio for other purpose.

Moreover, there are upper and lower bounds on the weight for all assets in the investment universe to ensure that neither of the following two extreme cases happen. First, there is only one position in the optimal portfolio with weight component equals to one which includes large idiosyncratic risks. Second, the final solution contains positions in nearly all risky assets, each of which has marginal weight proportion that is non-sensible. Investor is flexible to impose bounds that vary with each risky asset. For example, invest A shows special preference to AAPL stocks but relatively averse agains GM shares then he or she could choose a $[5\%, 80\%]$ bound for AAPL while a $[0\%, 2\%]$ bound for GM.

In general, weight components of each of the assets have to follow

$$l_{t,i}a_{t,i} \leq w_{t,i} \leq u_{t,i}a_{t,i}, \quad i = 1,\ldots,N \tag{3.15}$$

which is equivalent to

$$\begin{aligned} w_{t,i} \in [l_{t,i}a_{t,i}, u_{t,i}a_{t,i}] \quad & \text{if } a_{t,i} = 1 \\ w_{t,i} = 0 \quad & \text{if } a_{t,i} = 0 \end{aligned} \tag{3.16}$$

Equation 3.16 ensures that the optimal portfolio would either take significant amount in each of the positions in the portfolio and absolutely zero on the others. To put it in another way, the investor's wealth is fully allocated to the stocks in set In only.

3.3 The Complete Model Formulation

3.3.1 Fixed Transaction Cost CCMV Formulation

$$
\begin{aligned}
\underset{a_{t,i}, x_{t,i}, w_{t,i}, t_{t,i}}{\text{minimise}} \quad & \sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_{t,ij} w_{t,i} w_{t,j} \\
\text{subject to} \quad & \sum_{i=1}^{N} \mu_{t,i} w_{t,i} = R, \\
& \sum_{i=1}^{N} w_{t,i} = 1, \\
& a_{t,i} = \begin{cases} 1, & \text{if } x_{t,i} \geq 0 \\ 0, & \text{otherwise} \end{cases} \\
& \sum_{i=1}^{N} a_{t,i} = K, \\
& \sum_{i=1}^{N} |a_{t,i} - A_{t,i}| = \Delta, \\
& \sum_{i=1}^{N} fc_{t,i} |a_{t,i} - A_{t,i}| = TC_t, \\
& x_{t,i} = X_{t,i} + t_{t,i}, \\
& v_t = \sum_{i=1}^{N} P_{t,i} X_{t,i} - TC_t \\
& w_{t,i} = \frac{P_{t,i} x_{t,i}}{v_t}, \\
& l_{t,i} a_{t,i} \leq w_{t,i} \leq u_{t,i} a_{t,i}, \\
\text{where} \quad & i, j = 1, \ldots, N, \\
& \Delta = 0, 2, 4, \ldots, 2K.
\end{aligned}
$$

3.3.2 Linear Transaction Cost CCMV Formulation

$$
\begin{aligned}
\underset{a_{t,i},x_{t,i},w_{t,i},t_{t,i}}{\text{minimise}} \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{t,ij}w_{t,i}w_{t,j} \\
\text{subject to} \quad & \sum_{i=1}^{N}\mu_{t,i}w_{t,i} = R, \\
& \sum_{i=1}^{N} w_{t,i} = 1, \\
& a_{t,i} = \begin{cases} 1, & \text{if } x_{t,i} \geq 0 \\ 0, & \text{otherwise} \end{cases} \\
& \sum_{i=1}^{N} a_{t,i} = K, \\
& \sum_{i=1}^{N} |a_{t,i} - A_{t,i}| = \Delta, \\
& \sum_{i=1}^{N} lc_{t,i}P_{t,i}t_{t,i} = TC_t, \\
& x_{t,i} = X_{t,i} + t_{t,i}, \\
& v_t = \sum_{i=1}^{N} P_{t,i}X_{t,i} - TC_t \\
& w_{t,i} = \frac{P_{t,i}x_{t,i}}{v_t}, \\
& l_{t,i}a_{t,i} \leq w_{t,i} \leq u_{t,i}a_{t,i}, \\
\text{where} \quad & i,j = 1,\ldots,N, \\
& \Delta = 0,2,4,\ldots,2K.
\end{aligned}
$$

Notice that the only difference between these two models is the transaction cost function: the fixed transaction cost CCMV model for fixed hurdle commission fees and the linear one for those charged on per transaction amount basis. Please again refer to the list of Symbols following the table of content at the beginning of the paper for a complete collection of these symbols and notations that used in the models.

3.4 Computational Concerns

Although the two models built up above are both valid and theoretically soluble for powerful computational engines, in this section I would point out a number of straightforward facts underlying the models, as well as analyse some conditions and relaxations to release the computational burden and thus improving solving speed while not losing any legitimacy of the models.

1. The monetary value of the portfolio at time point T = 0 should be exactly the same as investor's initial wealth W_0. In other words, we set $V_0 = W_0$.

2. $A_{t,i}$, the binary variable representing whether asset i is currently included in the current portfolio or not at time point t, equals to 1 if it is, 0 otherwise is one of the intermediate results and is computed on demand. Mathematically,

$$A_{t,i} = \begin{cases} 1, & \text{if } i \text{ in } In \\ 0, & \text{otherwise} \end{cases}$$

3. In the model set up, we assume either fixed or variable transaction cost rate could be different for different risky assets at different time period. For simplicity reasons while not losing any generality, we impose the same cost rate for all risky assets over the whole investment period in the following analysis. That is to say, we assume there is a single fixed hurdle fee charged for all stocks in the asset universe and a single basis point rate on transaction prices for linear part of the costs that applied to each participants over the investment horizon.

4. Similarly, although we claim in the previous section that .both upper and lower bounds could vary from asset to asset and from time period to time period, we impose the same bounds on all the universe assets for the same reasons of simplicity and generality.

   Combining these two relaxations, we have

$$\begin{aligned} l_{t,i} &= l_{t,j} = l, \\ u_{t,i} &= u_{t,j} = l, \\ lc_{t,i} &= lc_{t,j} = lc, \\ fc_{t,i} &= fc_{t,j} = fc, \\ \text{for} \quad \forall i, j &= 1, \ldots, N, \\ t &= 0, 1, \ldots, T. \end{aligned}$$

Chapter 4

Results

Hence, we combine and test different optimization models and evaluate their performances mainly in two facets:

1. Static performance: Efficient Frontier
2. Dynamic performance: Rolling monetary portfolio value

4.1 Data Sets

I study and compare the performance of this newly developed CCMV model cooperating with transaction costs under different scenarios using latest market data for all 30 component stocks in the Dow Jones Industrial Average Index from Yahoo Finance. DJIA is widely agreed as **the** good representative benchmark of U.S. stock market.

To improve generality and rationality of the project, the analysis time horizon is chosen starting from March, 2008 to the latest data available time, which is March, 2017. I explicitly include the financial crisis time to study how the model performs in extreme cases. Figure 4.1 is a visual representation of the return distributions of asset returns, in the order of increasing standard deviation. It shows that nearly all of the 30 stocks has a sample mean close to zero with not large standard deviations. Thus it is reasonable to apply Mean Variance model on this data set.

Our optimization models with transaction costs were fed into computational engines and implemented using OPL, the IBM ILOG Optimization Programming Language, first published by The MIT Press in 1999, together with its script language. For the solver, I chose IBM ILOG CPLEX Optimization Studio which is a common solving software that searching for optimal solutions under Windows system . Moreover, the detailed scenario analyses and

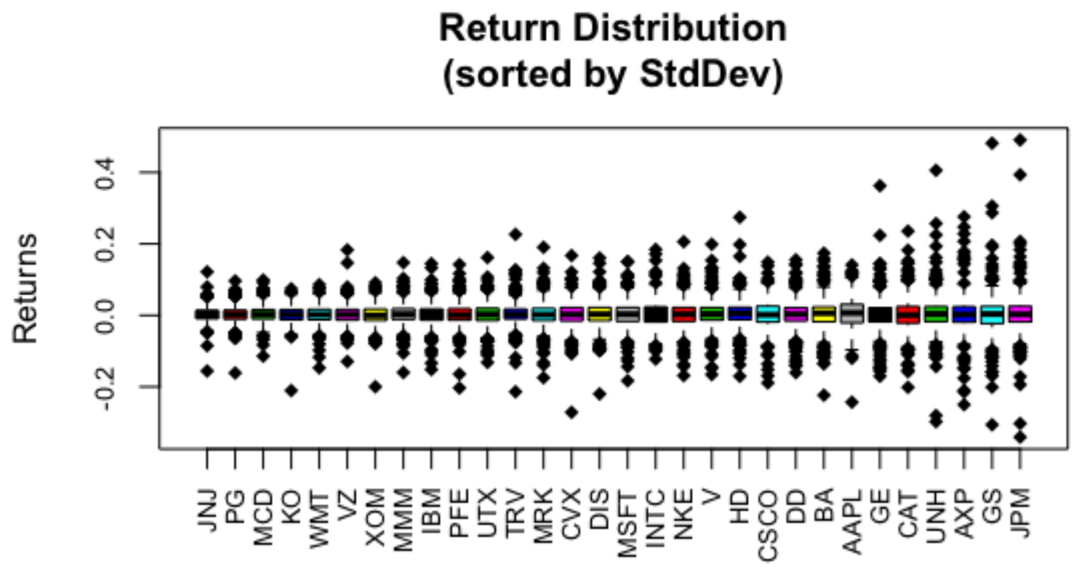


Fig. 4.1 Return distributions of DJIA components

recursive functions are written in R language and implemented in R studio. Here I point out the packages have been utilized throughout the project:

- SystematicInvestor
- PortfolioAnalytics package
- Portfolio Probe

which are some of the most widely used packages for financial applications in R.

4.2 Static performance: Efficient Frontier

First, to solve the Fixed Transaction Cost CCMV model, we consider a quarterly rebalanced portfolio with the asset universe as all N = 30 component stocks from the DJIA index. At each balancing time point t, we look back the most recent 3 months market prices, i.e. the latest quarter history of information and update the expected return, covariance matrix input for the model. Assume the desired number of distinct risky assets the investor wish to hold in the portfolio over the entire investment horizon is $K = 6$. We vary the desired number of changes in position in the new portfolio ranging from 0 to $2K$, in this case, 12, to study the effect of different position change levels on portfolio performances in the measure under

Mean- Variance framework. The results are presented in the form of mean-standard deviation diagrams as follows:

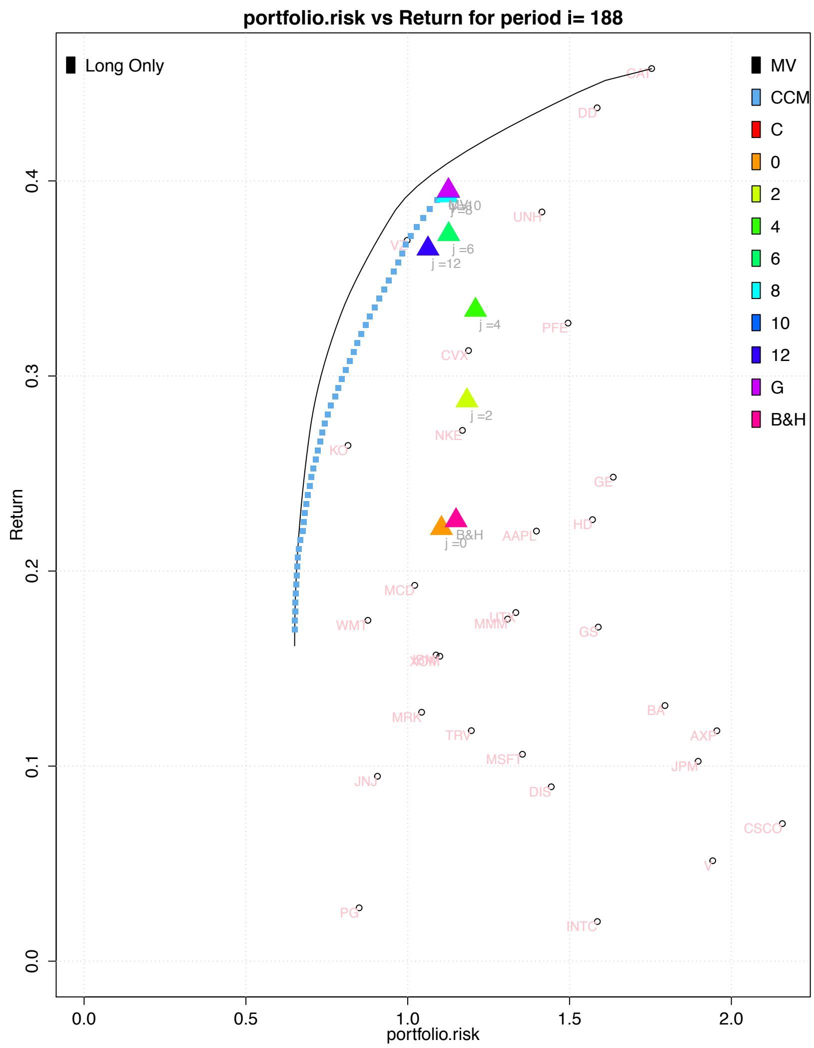
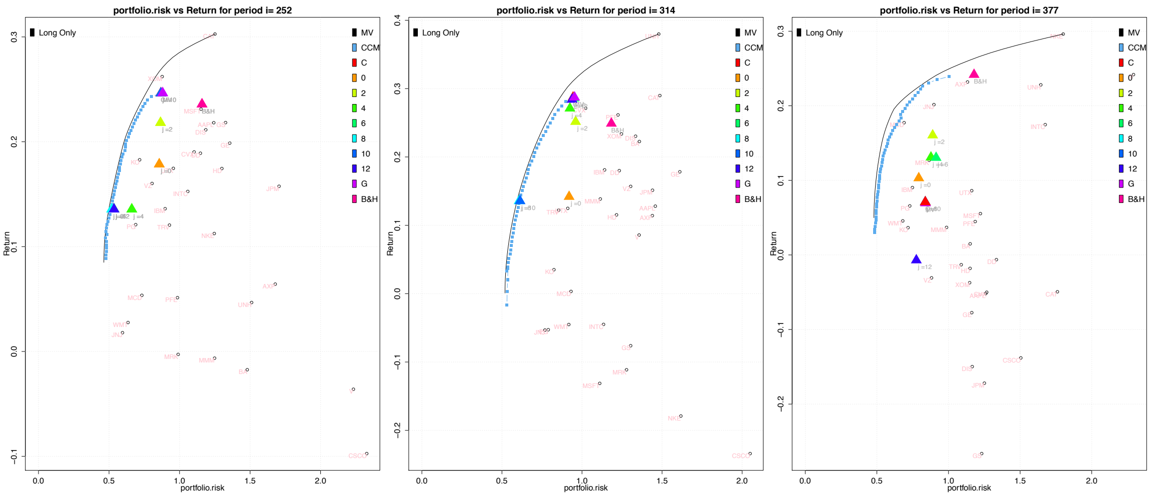


Fig. 4.2 Efficient Frontier for different at period i = 188

From the risk return diagram shown in figure 4.2, there are 12 labels at the top right corner explaining meanings of the scatters and frontiers. First, there are 30 risk return scatter points representing each of the expected performance of all the 30 DJIA component stocks, stock symbols are shown in pink color slightly beneath the corresponding points. Second, two efficient frontiers show in the middle of the graph, representing the best expected mean-standard deviation combinations for portfolios. The black dotted line represents efficient frontier for general long only Mean Variance model while the sky-blue dotted line shows that for cardinality model with no other constraints. Notice that since shorting is not allowed, the black curve terminates at the highest return point, which is CAT for the investment period

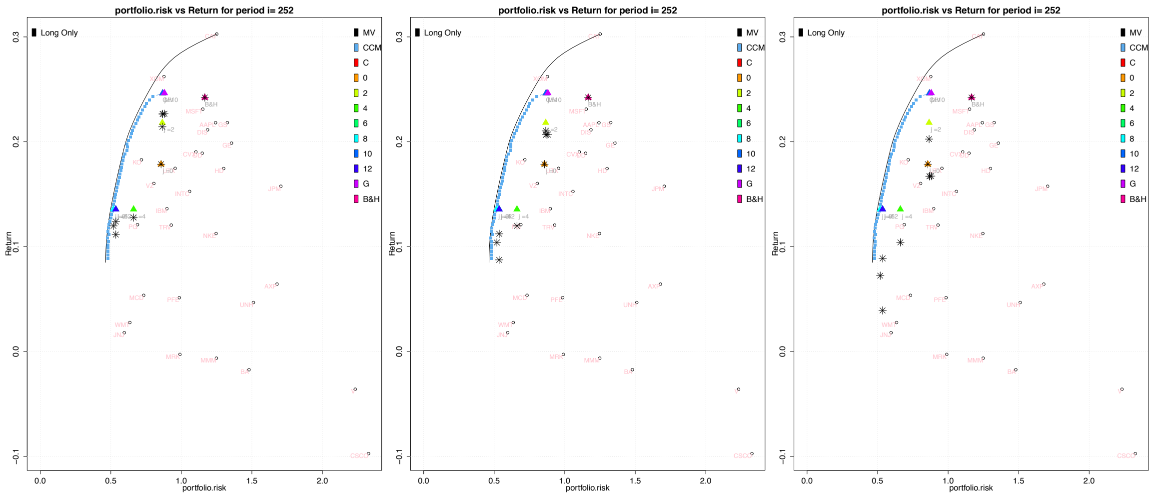


(a) i = 252 (b) i = 314 (c) i = 377

Fig. 4.3 Efficient Frontier comparison for different time periods i = 252, 314, 377 with no transaction costs

$i = 188$. It is sensible that the CCMV efficient frontier is under the MV efficient frontier since we restrict an extra constraint for CCMV and thus limits its behaviour. More importantly, there are 10 rainbow colored triangle shaped points representing 10 different the investor carries out for the investment period $i = 188$. Capital letter “C" denotes the scenario where the investor is rather indifferent to the transaction cost amount thus impose no position change constraint in the fixed CCMV model, whereas letter “G" represents the situation where the investor is even more indifferent about transaction fees thus further eliminate the cardinality constraint. “B H" shown in bright pink color is a common strategy usually known as “Buy and Hold". For this scenario, I first compute the optimal solution for a CCMV problem and then never do any changes in this portfolio. You may use the sentence "Buy and forget about it" to better understand this strategy. It is commonly computed as a benchmark to better study the active investment. The remaining 9 labelled as even numbers ranging from 0 to 12 just represents the RHS of constraint 3.4 which is the desired level of position changes Δ. You many wonder the difference between the strategy of "Buy and Hold" and the "zero position change" strategy, which looks similar at the first thought, but indeed they are different, in the sense that the proportion in the "zero position change" strategy may vary from time to time, while that in "Buy and Hold" is remain unchanged.

Here are the remaining plots in this series of figures sharing the same structure and purpose only with different expected return, covariance matrix and market prices input. Observe, compare and contrast them provides me with several interesting findings and further enlightens a number of investment insights.



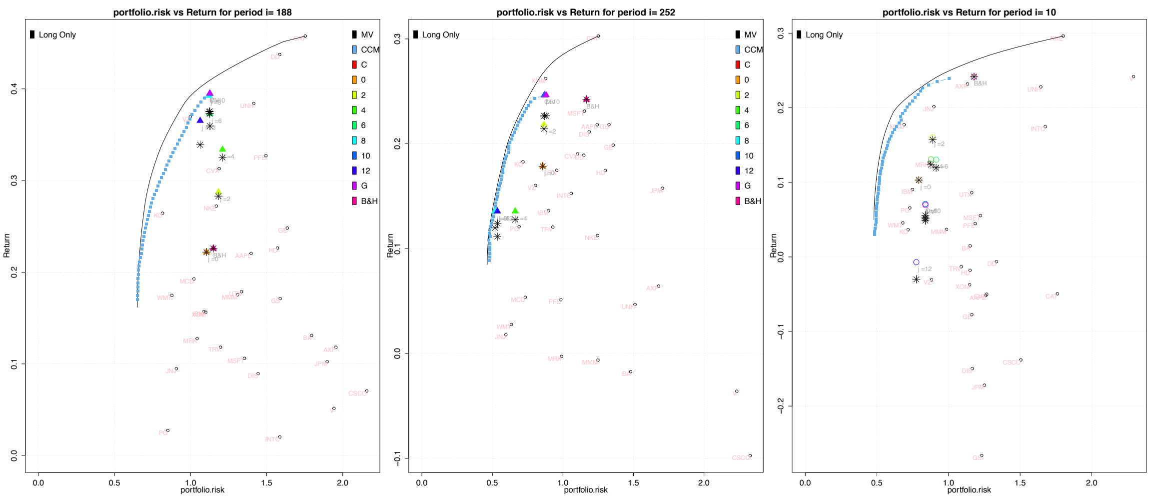
(a) \$2.5 per position change (b) \$5 per position change (c) \$10 per position change

Fig. 4.4 Efficient Frontier for different at period i = 252 with fixed transaction costs

Figure 4.3 gives a comparison among various efficient frontiers at different time periods, with t = 252, 314, 377 respectively. Since the expected return, covariance matrix and market prices input varies from each other, the scatter plots of each risky asset i and the correspondent efficient frontiers differ too, so are the performances of our 10 . Yet, there are some general findings for efficient frontiers with no transaction costs summarized as below:

- The CCMV efficient frontiers closely reaches around the MV efficient frontiers for all of the three time periods. This phenomenon makes sense since shorting is not allowed in the MV model here, many of the weight components are automatically forced to be zero. Hence, CCMV efficient frontiers do not differ a lot from MV efficient frontiers.
- The performance of various strategies depends on the specific investment settings and the assessment of goodness of largely depends on the investor's risk attitude . For example, the portfolio that setting position change as 10 has the highest expected return at period i = 252, whereas "Buy and Hold" strategy is the best in terms of expected return at period i = 377. In addition, at each time periods, some could realize the best theoretical performance as they almost touches the efficient frontier.

Figure 4.4 and 4.5 give a nice visual presentation of the effect of transaction costs on the portfolio performance. The net in transaction cost portfolio performances are presented by black asterisks. In detail, the sub plots in figure 4.4 share the same background composed of individual scatters and the two efficient frontiers because they share the same set of data input at period i = 252. In addition, the rainbow colored strategy performance in prior to the transactions also remain the same for these three. The key point worth attention here,



(a) i = 188 (b) i = 252 (c) i = 377

Fig. 4.5 Efficient Frontier comparison for different time periods i = 188, 252, 377 with fixed transaction cost \$2.5 per position change

however, is the relative positions of the black asterisks compared with the triangles. For both “Buy and Hold" and “$\Delta = 0$" , no matter how large the fixed transaction costs rates are, they remain the same. The reasons lie in the very strategy nature as there would be no position changes for these two portfolios throughout the entire investment horizon. While for others, larger the position change Δ, more obvious the effect of fixed transaction rates is. In the last subplot, For example in the last subplot, \$10 per position change transaction rate drops the expected return of strategy: “$\Delta = 12$" more than half to only about 0.045%. And the effect is also strategy-dependent since “$\Delta = 12$" has the most rapid position changes . To conclude, the effect of transaction costs on efficient frontier mainly appears to be the reduction on the portfolio expected return while the portfolio risk remains unchanged. Higher the position change level, more rapid the transaction activities, thus larger the transaction fees and more reduction on the expected alpha.

Figure 4.4 compares the efficient frontiers from another angle and reveals how effects differ for different time periods imposing the same rate of transaction cost. Usually the red color triangle, representing the “cardinality" portfolio is unseen from the plot simply because this is covered by the same performance scatter of the position change strategy. Each cardinality solution itself would also have a similar position change vector, not as a constraint input , but an intermediate result, which would be coincided with the same position change level constrained model.

4.3 Dynamic performance: Rolling monetary portfolio value over investment period

Note: *Please explore the interactive versions of these figures for the best presentation and observation of this section by clicking the light-blue hyperlink given right after each static figures.*

To better study the model performance, I build up 10 by combining different parameters to make comparisons under different scenarios that may appeal to different investors. In the order that consistent with that of the labels shown in figure 4.6 dynamic, I now explain the portfolio settings of each of the 10 :

1. Mean variance Portfolio Selection model in-cooperating with transaction costs.
2. Mean variance Portfolio Selection model.
3. Cardinality constrained Mean variance Portfolio Selection model.
4. Cardinality constrained Mean variance Portfolio Selection model in-cooperating with transaction costs.
5. Mean variance Portfolio Selection model with no weight bounds constraints. One more words here, since there are bounds on weight vectors in the model described in Chapter 3, this strategy is deliberately carried out to compare with the performance of opMV to study how weight bounds would affect the final solution and if so, what are the magnitude.
6. The next five model are the replication of the above ones with investment capital updated by the transaction costs incurred. To put it in another way, no matter whether transaction costs are computed in the previous or not , those five models assume the amount of transaction costs are negligible thus does not take it into consideration for the evolution of monetary value. Whereas all the with "withTC" suffix are the corresponding with portfolio value updated in the procedure described in section 3.2.5

All of the strategies start with initial monetary value \$1000,000, that is $W_0 = V_0 = \$1000,000$. And the investor decides a cardinality level $K = 6$ for all the RHS of the cardinality constraints 3.3 equation meaning that as long as the involving the decision of selecting K risky securities from the investment universe, in this case, it is the 30 risky component stocks of DJIA Index, he or she would maintain the portfolio size to be 6 over the entire time period. The investor put 2.5% lower bounds and 20% upper bounds on all of the positions.

We assume several set of parameters in the transaction cost functions:

- Fixed cost rate: \$5 per position change;
- Fixed cost rate: \$100 per position change;
- Linear cost rate: \$10 basis point of the transaction amount.

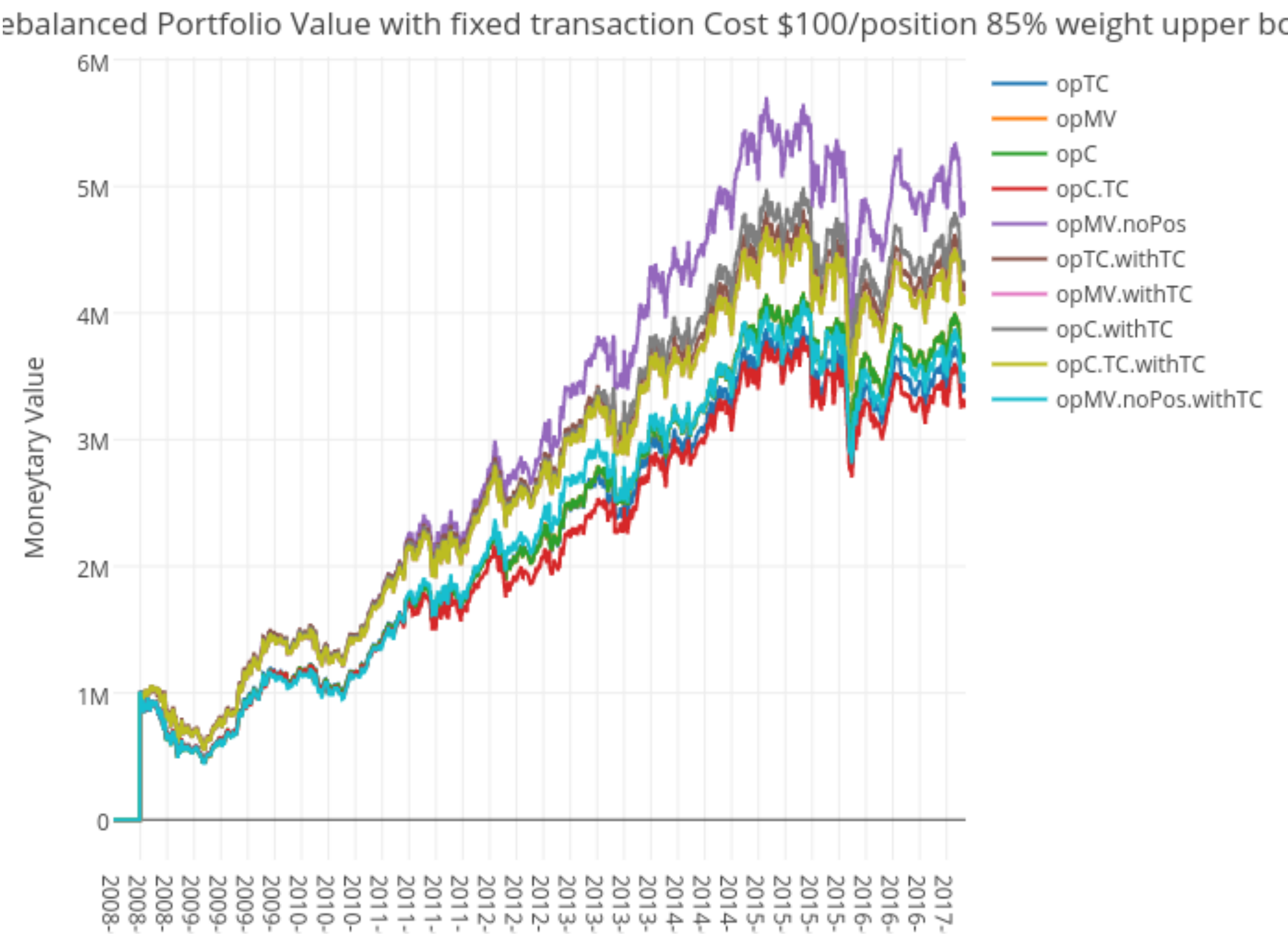


Fig. 4.6 monetary value of a quarterly rebalanced portfolio with fixed transaction cost constraint

I will first elaborate on figure 4.6 *dynamic* which shares the similar structure with all of the other figures to be shown in the following. Here are some observations and their implications:

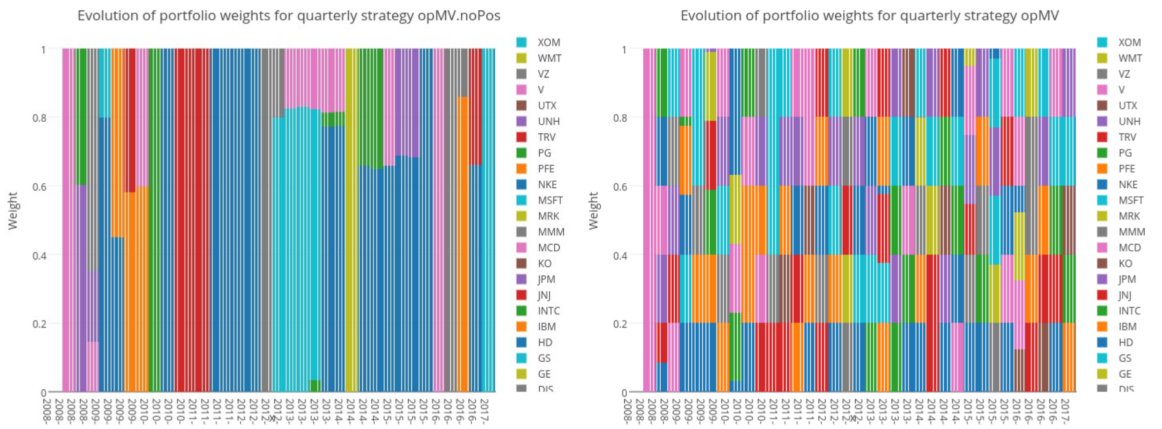
- First, this figure serves as a powerful evidence against the frictionless assumption in MV model. Each of the former without "*withTC*"-suffix, representing portfolios in perfect and frictionless world, always outperforms its corresponding "*withTC*"-suffix counter-party. This is quite understandable: investors suffer from transaction costs in the real life, which implies that we shall never be naive to neglect transaction costs in the frictional investment settings in real life. Those theoretical portfolio monetary growth are too good to be true;

- Secondly, if we take SP 500 as a benchmark which has a performance of no more than 150% accumulated return, the strategies shown in above had bravo performances in terms of realized return which is due to positive asset allocation.

- In general, these portfolio show similar evolution trends since they are all under the mean variance framework. All of the 10 suffer a deep loss during the financial crisis period but recover year by year during the total 10 years time window ;

- Now focusing only on the *opMV* (Mean variance Portfolio Selection model) and *opMV.noPos* (Mean variance Portfolio Selection model with no weight bounds constraints) . Observing that the purple trend is above the orange line over the entire 10 years, meaning that imposing extra weight constraints do limit a lot of the portfolio performance. However, for the reasons explained in section 3.2.7, we have to inject this constraint in out model to ensure diversification and rationality. To investigate further, we compute stacked bar-plot representing the evolution process of weight vectors over the investment horizon for these two different and it is shown in figure 4.7. In this figure, x-axis represents the time series and y-axis denotes the weight components of the portfolio. Different colors represents different asset holdings in the portfolio at that time period. More the colored blocks, larger number of risky assets are hold in that portfolio. Noticing that in the left sub-panel, there are only minimal number of weight components for strategy *opMV.noPos*. During most of the 10 years, there are only one or two, or up to three different color bars made up the vertical 100% fully invested weight constraint, meaning that the investor's capital is only allocated to very few amount of risky assets which would involve large idiosyncratic risks. Compared with that in the right panel for strategy *opMV*, with bounds imposed on the weight vectors, there are at least 5 different colored bars at each of the investment time point. Recall this is because the upper bound for all of the assets are 20%.

- Now let us turn attention to CCMV models which are the focus of this paper. Observing that in figure 4.6 the green *opC* approximate closely with the orange *opMV*



(a) opMV.noPos Weight Evolution

(b) opMV Weight Evolution

Fig. 4.7 Weight Evolution comparison: opMV v.s. opMV.noPos

and one could hardly note the differences between them. While for the trends with transaction costs, one can easily distinguish between the pink *opMV.withTC* and the gray *opC.withTC*. Possible reason behind this phenomenon could lie in the fact that with a cardinality constraint, there are limits on open positions in the portfolio, which then incur less transaction costs. To verify this conjecture, we made more explorations on the phenomenon . In figure 4.8, we aim to study the effect of transaction costs on various different strategies. The pair of time series sharing the same color represent the pair of with and without the update of transaction costs. It is clear that the transaction costs have the much smaller effect on with cardinality constraints. Drops are nearly non-significant for both the lime-blue *opC* and purple *opC.TC* which consolidate our reasoning that imposing cardinality constraints force less position changes along the investment period which then results in less amount of transaction costs. This finding is exciting and evolutionary because theoretically, MV would always outperforms CCMV model, however here after incorporating the transaction costs, there is a complete reversal of the story.

Up to this point, I could new tell the purpose of building such a cardinality constrained mean variance Portfolio Selection model in-cooperating with transaction costs, namely *opC.TC*. Compare the lime-blue *opC* and purple *opC.TC* time series in figure 4.8. Once cardinality constraints are already present in the model, adding transaction costs constraints does not change the result a lot, for the same reasons analyzed above.

This motivates me to think deep about the model. We create cardinality constraints and in-cooperate it into portfolio optimization model in the aim of avoiding large number of transaction fees. Only imposing transaction costs constraints by no means ensure the

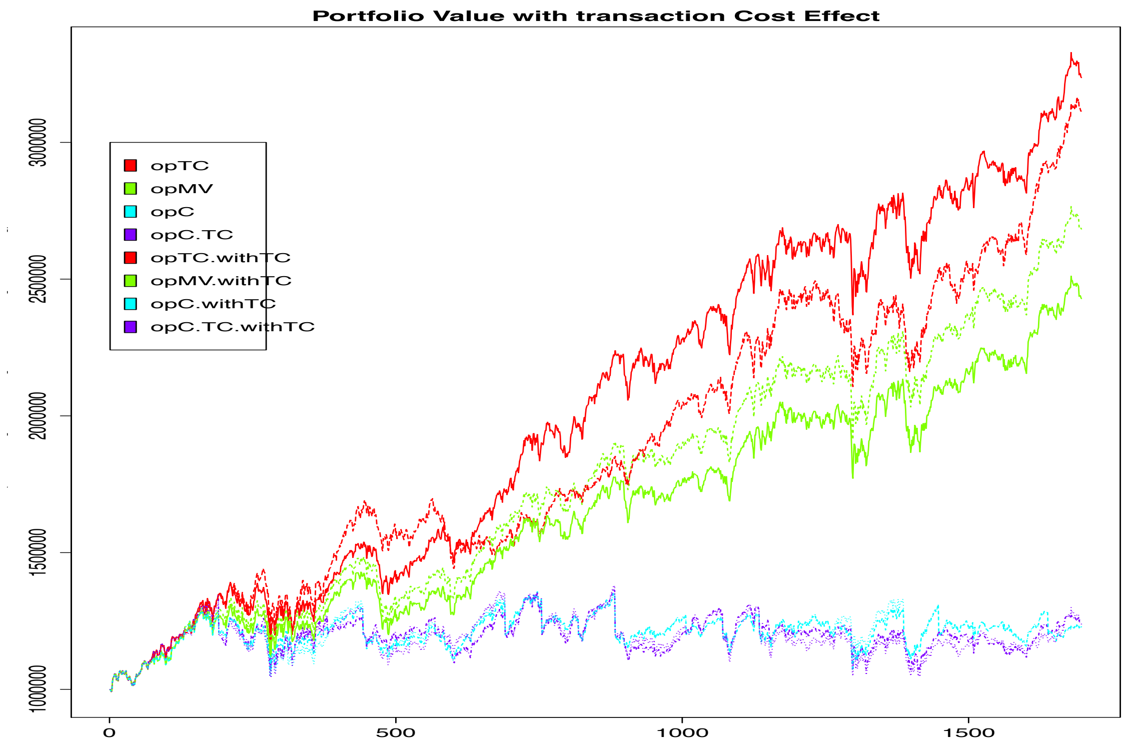


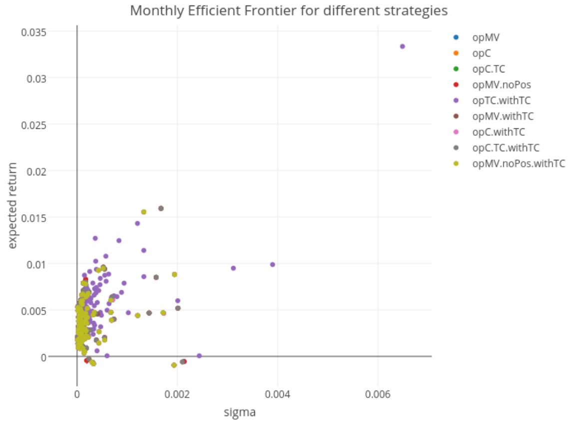
Fig. 4.8 Focused Monetary portfolio value over one single investment period

desired level of portfolio size while it is not the same the other way round: imposing only cardinality constraints on the portfolio optimization model not only restricts the portfolio size to be exactly the number of positions the investor would like to hold, but it simultaneously prevents extreme turnovers or large amount of transactions. This is indeed the advantage of cardinality constraint over the other side constraints such as transaction cost constraints and turnover constraints.

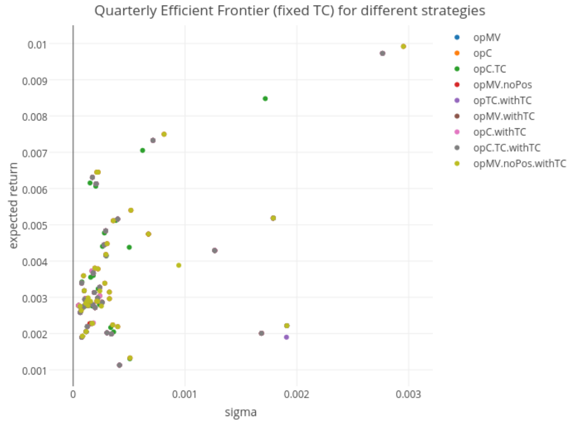
4.3.1 Re-balancing frequency: Monthly v.s. Quarterly

In this part we are particularly interested in the effect of re-balancing frequency on the portfolio performance. Consider two investment only differ in re-balancing frequency, one of them is monthly rebalanced while the other is quarterly rebalanced, all other portfolio settings are the same. Namely, in the first strategy, the investor is impatient and examines his or her portfolio on a monthly basis and would change his portfolio holdings if necessary while the later investor check the portfolio per three months. Here are the differences.

In figure 4.9 *dynamic (a)*, *dynamic (b)* I collect the realized portfolio mean- standard deviation data pairs and map them to the risk return diagram, in the attempt of reproducing the shape of efficient frontiers in MV model. There are 107 data samples for each of the 10

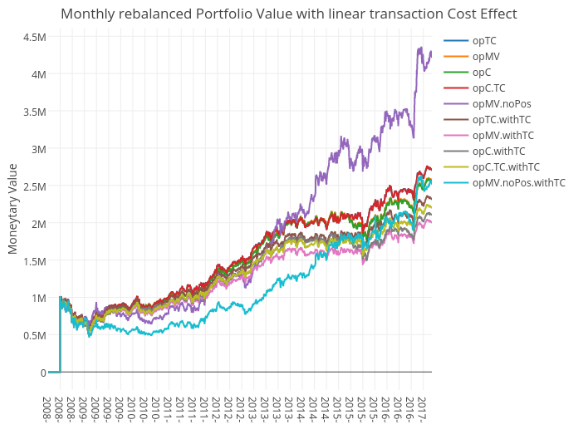


(a) Monthly Efficient Frontier

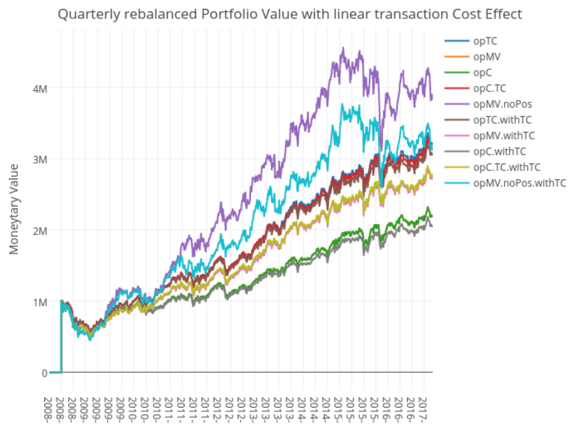


(b) Quarterly Efficient Frontier

Fig. 4.9 Efficient Frontier comparison: Monthly v.s. Quarterly, linear cost



(a) Monthly linear cost

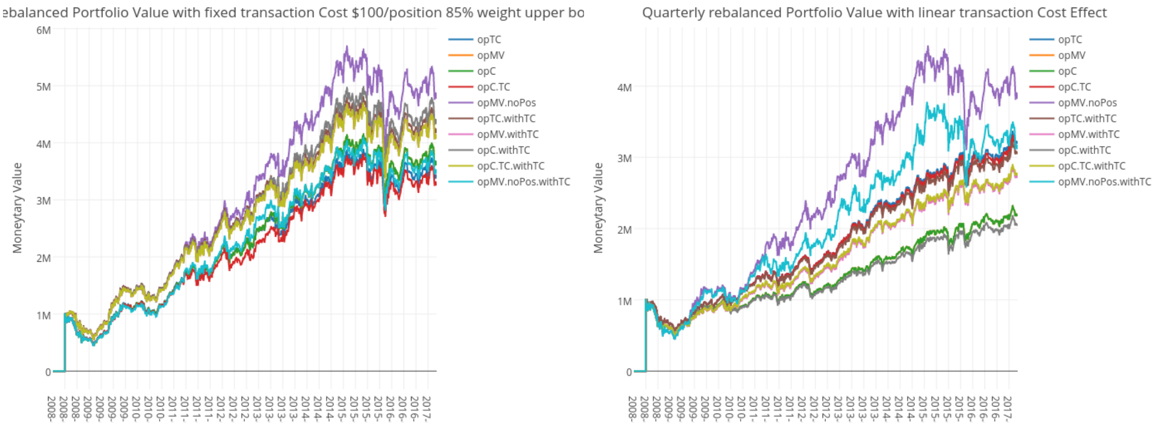


(b) Quarterly linear cost

Fig. 4.10 Monetary portfolio value comparison: Monthly v.s. Quarterly, linear cost

for monthly rebalanced portfolios and 37 for the quarterly rebalanced ones. So in total there are $107 \times 10 = 1070$ and $37 \times 10 = 370$ scatters in the left and right panel respectively. One is easier to imagine the "Boomerang" shape of efficient frontier from the monthly sub-plot but harder for the right sub-plot due to data deficiency. We find that large number of points concentrate together within the low risk low return region (lower left corner) but it is hard to derive any conclusion as the realized data plots could end up realized in anywhere among the risk return diagram

Next, we compare these two portfolios in terms of monetary portfolio value shown in figure 4.10 *dynamic a*, *dynamic b*. Compare the left and right panel, one can conclude that frequent re-balancing one's portfolio worsens the reduction of transaction fees on the portfolio value. For example, focus on the *opMV.noPos* and *opMV.noPos.withTC* time series on the two



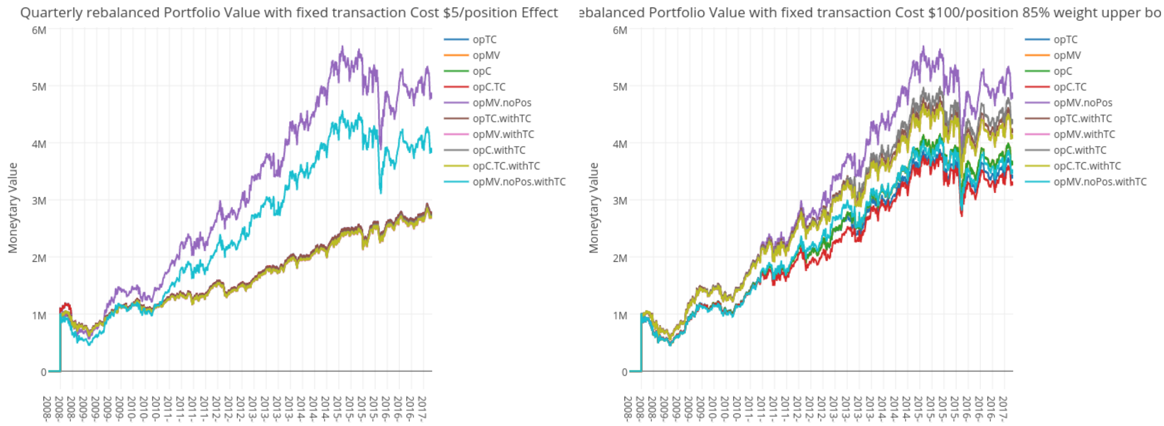
(a) Quarterly fixed cost (b) Quarterly linear cost

Fig. 4.11 Transaction Cost type: Linear v.s. Fixed

sub plots, the value of *opMV.noPos.withTC* is only around half of the value of *opMV.noPos* for the major time periods on the left. Although the value of *opMV.noPos.withTC* is also below that of *opMV.noPos*, the difference is much smaller for the quarterly rebalanced portfolio. This suggests that frequently adjust one's portfolio incurs large amount of transaction costs, which would even eat up most of the investor's capital for small investors. However, there are also benefits. Focusing on the time periods of two financial turmoils at the end of 2008 and in the beginning of 2016, the monthly balanced portfolio shows more stable performance than the quarterly one due to its diligence on frequently checking on the market thus avoiding extreme losses caused by the crisis.

4.3.2 Transaction Cost type: Linear v.s. Fixed

In the following we are going to analyze the impact of different transaction costs of the optimal results. Figure 4.11 *quarterly fixed 100*, *quarterly linear* compares portfolio growth for a quarterly rebalanced portfolio under different transaction cost functions. The left figure shows a realization of fixed transaction cost at $100 per position change while the right one represents a $10 basis point commission charge on per dollar of the transaction amount. Once again there are no solid conclusions about this group of comparison. Generally, we observe that the one-to-one correspondence between various and their with transaction cost counter-parties are more obvious for the linear type of transaction costs due to the existing linearity.



(a) Quarterly fixed cost $5 per position (b) Quarterly fixed cost $100 per position

Fig. 4.12 Transaction Cost effects, magnitude: $5 v.s $100

4.3.3 Transaction Cost Magnitude Effects

In this last part of the section, I present the impact of the magnitude of fixed transaction cost rates. figure 4.12 *dynamic 5*, *dynamic 100*. The largest findings from this group of comparison is that the rate of $5 per position being relatively subtle compared with the large portfolio capital in million dollar scale, the drops of “with.TC" are also small compared with that of the $ 100 magnitude.

Chapter 5

Conclusion

5.1 Summary

The aim of this thesis was to contribute to the development of cardinality constrained portfolio optimization models with transaction cost.

In Chapter 2 we gave a review of previous studies on cardinality constrained portfolio optimisation with transaction cost. Work on cardinality constraints mean variance model in the last two decades has been largely focused on developing heuristic algorithms with main efforts paid on imposing side constraints such as turnover constraints and number of trades constraints, none of which could guarantee the existence of an optimal solution. For the work on transaction cost, there seems no consensus agreed upon the formulation of transaction cost functions, making studies independent and disjointed with each other.

Given the fact that there exists no exact re-balancing (dynamic) algorithms to CCMV with transaction costs problems according to the literature review, the whole modified CCMV net in transaction costs effect models fully described in Chapter 3 are original works and completely brand-new. Especially for the formulation of position change constraint, there could be even more applications for real life problems in many aspects which is elaborated in the following section. Chapter 3, where we presented optimal solutions for the transaction cost model, contains the first original work in this thesis. We began by giving our model a detailed problem setup, explaining the typical problems encountered in real life and give out the objective function. I then go through each of the constraints appeared in the model for their reasons for existence, underlying logic and computational methods. Tables, figures and simple illustrating examples are included for better understanding. We then integrate the model setups and give the complete model formulations for both fixed and linear constraints. In the last part, we point out several practical concerns and give the corresponding relaxation conditions.

To showcase the performance of my model, figures of efficient frontiers and of rolling monetary value of the portfolio are given in the previous chapter, chapter 4, for static and dynamic performance respectively.

The experience is exciting and evolutionary because theoretically, Mean Variance models would always outperforms CCMV models with cardinality constraints, however here after incorporating the transaction costs, there is a complete reversal of the story. From both intuition and the efficient frontier plots given in figure 4.2, the black solid efficient frontier for MV model is always on the above of the dark blue dotted curve, which represents the most efficient portfolio realizations for CCMV models. This is due to the fact that imposing cardinality constraint limit a smaller asset universe from which the optimal solutions are drawn. Hence, from the very formulation of the model, MV would always be better for investors, guarantees a higher expected return for the same level of risk or provides smaller risk exposure at the same targeted expected return level. However, incorporating with the transaction costs, the net in transaction cost effect portfolio risk-return tradeoff are much more complex in reality.

5.2 Further Applications of the Position Change Constraint

This model could have a lot of applications, to name a few, consider the following examples.

In a class of a secondary school of size n, say 50, we select a subgroup of *elite stream* with only size N, say 6, how to choose initially these 6 team members is a cardinality constrained problem. Yet, how to balance and maintain the team members could be solved by this CCMV model incorporating position change constraint. There are usually a number of practical objectives on the implementation of such an *elite stream*:

1. Stability.

2. Competitiveness.

The first requirement ensures a degree of sense of security for students currently within the group, which in turn is how it attracts those currently out of the group; while the second requirement could be regarded to facilitate competitiveness and thus increasing the average performance of the whole class. Our model is effective and efficient in achieving both of the two objectives.

More specifically, we substitute the expected return in the portfolio optimization problem by each student's expected overall score and replace the covariance matrix by the progress rate representing the improvement potential of the student. Then the model could be interpreted as to solve the following problem: given each student's expected overall score and potential progress rate, what will be the optimal evolution of team member components for a desired time interval, for example, one academic year with monthly examination, i.e. updates in the aim of maintaining both the stability of the team and increasing the average performance of the whole class by facilitating competitiveness?

Such an “Elite stream design" is rational and has a wide-spread use in both academic and industrial organizations. Increasing the RHS of the position limit Δ would result in more intense competitiveness thus picking an appropriate number K really depends on the manager’s preference between stability and competitiveness. And it is only one of the many applications, other examples would be: customer loyalty group, balance of daily in-taken calories, managing life activities..., which will be skipped in this thesis for simplicity reasons.

5.3 Model evaluation

5.3.1 Advances of the model

One could probably sense how flexible and customized the model can be by reading through the previous chapters. To emphasize here, investors are not only able to set the general parameters of their portfolios such as rebalacing frequency, desired position changes Δ and preferred portfolio size K, but could also fine tune on the detail settings such as weight bounds on each of the risky assets.

5.3.2 Further Developments

Despite the various advantages of the model, it is still deficient in the following considerations which I feel obliged to put it frankly for the sake of the whole research community and investment society . Here in the model we generally assume that investors would choose certain strategy and stick to it for the entire investment horizon. In other words, although the dynamic model do gives optimal solutions according to different time periods, the investor is not allowed to pick multiple position change preferences during the investment horizon. A more logical, flexible and profitable strategy would be like this: at each rebalancing time point, the investor could also modify the strategy he or she wants to use according to the latest market condition and clients’ capability. For example, investor would like to pick zero position change or even “buy and hold" strategy when there are large economic turmoil while choose to hole general unconstrained portfolio when large amount of capital is injected to the original investment. However, such a problem makes Δ no longer a constant but a random variable. It is NP-hard already not to mention developing a dynamic investment strategy policy! But solving that challenging yet exciting problem would be highly profitable. For example, we could extend the original "Long-only" portfolio by relaxing the weight constraints to increase profitability, possible solutions may be building a 130/30 index as suggested by Prof. Andrew Lo [6].

To conclude, sincerely hope that all the original contributions to knowledge in this thesis could be appreciated and I will keep researching on the remaining problem to bring even more efforts.

References

[1] Baule, R. (2010). Optimal portfolio selection for the small investor considering risk and transaction costs. *OR spectrum*, 32(1):61–76.

[2] Crama, Y. and Schyns, M. (2003). Simulated annealing for complex portfolio selection problems. *European Journal of operational research*, 150(3):546–571.

[3] Elton, E. J. and Gruber, M. J. (1977). Risk reduction and portfolio size: An analytical solution. *The Journal of Business*, 50(4):415–437.

[4] Jiang, K., Li, D., Gao, J., and YU, J. X. (2014). Factor model based clustering approach for cardinality constrained portfolio selection. *IFAC Proceedings Volumes*, 47(3):10713–10718.

[5] Li, D., Sun, X., and Wang, J. (2006). Optimal lot solution to cardinality constrained mean–variance formulation for portfolio selection. *Mathematical Finance*, 16(1):83–101.

[6] Lo, A. W. and Patel, P. N. (2008). 130/30: The new long-only. *The Journal of Portfolio Management*, 34(2):12–38.

[7] Lobo, M. S., Fazel, M., and Boyd, S. (2007). Portfolio optimization with linear and fixed transaction costs. *Annals of Operations Research*, 152(1):341–365.

[8] Mansini, R., ,odzimierz Ogryczak, W., Speranza, M. G., and of European Operational Research Societies, E. T. A. (2015). *Linear and mixed integer programming for portfolio optimization*. Springer.

[9] Reeves, C. R., Rayward-Smith, V., Osman, I., and Smith, G. (1996). Modern heuristic search methods. *Editorial Jon Wiley and sons. Chichester, Inglaterra.*

[10] Satchell, S. and Scowcroft, A. (2003). *Advances in portfolio construction and implementation*. Butterworth-Heinemann.

[11] Soleimani, H., Golmakani, H. R., and Salimi, M. H. (2009). Markowitz-based portfolio selection with minimum transaction lots, cardinality constraints and regarding sector capitalization using genetic algorithm. *Expert Syst. Appl.*, 36(3):5058–5063.

[12] Website (2017a). Commissions. https://www.tradeking.com/rates. [Online; accessed 17th April, 2017].

[13] Website (2017b). Transaction Costs. https://www.hkex.com.hk/eng/market/sec_tradinfo/trancost.htm. [Online; accessed 17th April, 2017].

[14] Woodside-Oriakhi, M. (2011). *Portfolio optimisation with transaction cost*. PhD thesis, Brunel University, School of Information Systems, Computing and Mathematics.