Dynamic Cardinality Constrained Portfolio Optimization

with fixed and linear Transaction Costs

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Outline

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A confused investor. . .

Assume an investor, say April, would like to invest in stock markets. Having advanced knowledge about Mean Variance Model, she would like to attain an efficient portfolio as the market portfolio.

Yet, she only has a **small** amount of initial capital.

She would also like to maintain the portfolio over **long** time periods.

She could . . .

- Buy and Hold

She could . . .

- Buy and Hold
- Believe in theoretical model

She could ...

- Buy and Hold
- Believe in theoretical model
- Being more realistic and sensitive

DYNAMIC REBALANCING MODEL

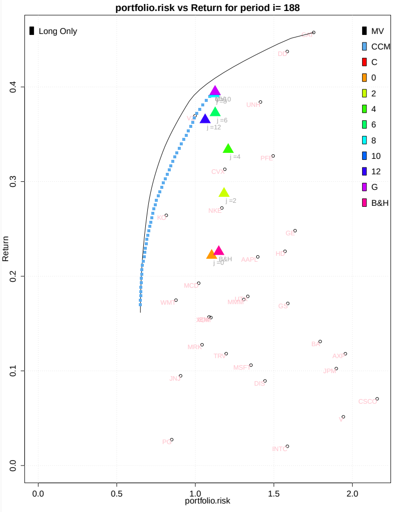

Figure 1: Efficient Frontier for different at period i = 188

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Symbols and Notations i

Parameters

N: the number of asset classes in the assets universe

f: portfolio rebalancing frequency, could be daily, monthly, quarterly or yearly

K: the desired number of distinct risky assets to hold

W_0: initial wealth, also the initial investment capital to start investment

Symbols and Notations ii

Input in prior to rebalancing time point t

$\mu_{t,i}$: the expected return vector

$\sigma_{t,ij}$: the covariance matrix

$P_{t,i}$: the current market price of per share of asset i at time point t

$c_{t,i}$: the transaction cost rate if any trading of asset i is incuured, $c_{t,i} = fc_{t,i}$ for fixed transaction cost and $c_{t,i} = lc_{t,i}$ for linear cost type

$l_{t,i}$: lower bound on portfolio weight

$u_{t,i}$: upper bound on portfolio weight

$X_{t,i}$: the number of shares hold in the current portfolio of asset i (i = 1,...,N) at time point t

Symbols and Notations iii

Intermediary results for current portfolio

$A_{t,i}$: binary variable representing whether asset i is currently included in the current portfolio or not

$W_{t,i}$: portfolio weight vector of asset i in the current portfolio

Symbols and Notations iv

Decision variables for investment period [t, t+1]

$x_{t,i}$: the number of shares to hold in the portfolio of asset i in the new portfolio at time point t

$a_{t,i}$: binary variable representing whether asset i is to be included in the new portfolio at time point t, equals to 1 if it is, 0 otherwise

$t_{t,i}$: the number of shares to trade in the position on asset i in order to get the optimal portfolio at time point t

$w_{t,i}$: portfolio weight vector of asset i in the new portfolio

The Complete Model Formulation

$$
\begin{aligned}
\underset{a_{t,i},x_{t,i},w_{t,i},t_{t,i}}{\text{minimise}} \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{t,ij}w_{t,i}w_{t,j} \\
\text{subject to} \quad & \sum_{i=1}^{N}\mu_{t,i}w_{t,i} = R, \\
& a_{t,i} = \begin{cases} 1, & \text{if } x_{t,i} \geqslant 0 \\ 0, & \text{otherwise} \end{cases} \\
& \sum_{i=1}^{N} a_{t,i} = K,
\end{aligned}
$$

The Complete Model Formulation (continued)

$$
\begin{aligned}
\text{subject to} \qquad & x_{t,i} = X_{t,i} + t_{t,i}, \\
& w_{t,i} = \frac{P_{t,i} x_{t,i}}{v_t}, \\
& v_t = \sum_{i=1}^{N} P_{t,i} X_{t,i} - TC_t, \\
& \sum_{i=1}^{N} w_{t,i} = 1, \\
& l_{t,i} a_{t,i} \leqslant w_{t,i} \leqslant u_{t,i} a_{t,i}, \\
\text{where} \qquad & i, j = 1, \ldots, N.
\end{aligned}
$$

Objective Function

$$\underset{a_{t,i},x_{t,i},w_{t,i},t_{t,i}}{\text{minimise}} \quad \sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_{t,ij} w_{t,i} w_{t,j}$$

Observations:

- in **consistent** with the original Markowitz model: minimizing volatility
- tiny yet vital **differences**: formulation of $w_{t,i}$

Formulation of $w_{t,i}$ and Balance of portfolio value

$$w_{t,i} = \frac{P_{t,i} x_{t,i}}{v_t}, \quad (1)$$

where

$$V_t = \sum_{i=1}^{N} P_{t,i} X_{t,i}$$

$$v_t = V_t - TC_t$$

$$x_{t,i} = X_{t,i} + t_{t,i}, \quad i = 1, \ldots, N$$

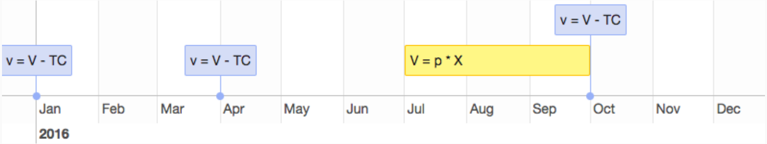


Figure 2: Evolution of a quarterly rebalanced portfolio value

Transaction Cost

Multiple Transaction Cost Models [2]

1. linear cost model: fixed cost per incremental trade
2. fixed hurdle(threshold) model: fixed hurdle cost incurred for making the trade
3. with higher orders (quadratic,. . .)

Transaction Cost

In practice, the former two are more common.

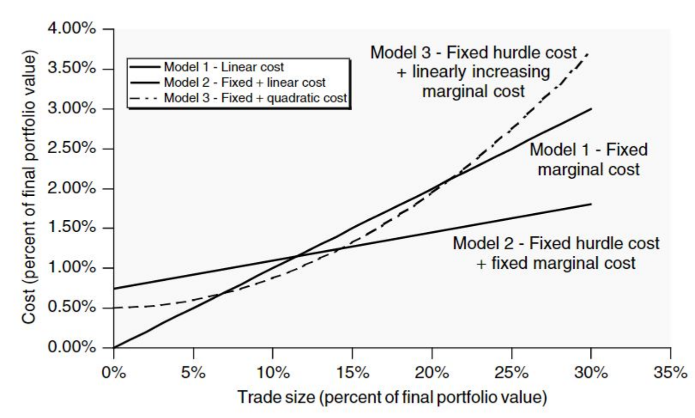


Figure 3: Common Transaction Cost Models [2]

Position Change Constraint

$$\sum_{i=1}^{N} |a_{t,i} - A_{t,i}| = \Delta \tag{2}$$

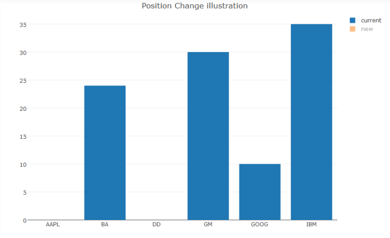
where

$$\Delta = 0, 2, 4, \ldots, 2K \tag{3}$$

Table 1: Example to illustrate how position change constraint works

position index	AAPL	BA	GM	IBM	DD	GOOG
current position index	0	1	0	1	1	1
new position index	1	1	0	0	1	1

Illustration



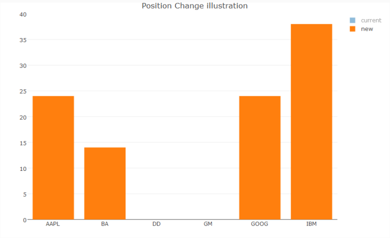


Illustration Example: Current and new positions in a simple portfolio

Another interpretation by SET

It may be easier to understand by imagining two sets to represent the whole risky assets universe. Let

$$In = \{\, i \mid \text{risky asset i is currently in the portfolio} \,\} \tag{4}$$

and

$$Out = \{\, i \mid \text{ asset i is not in the current portfolio} \,\} \tag{5}$$

Then, for this example,

$$In = \{\text{BA, GM, GOOG, IBM}\}$$

$$Out = \{\text{AAPL, DD}\}$$

Fixed Transaction Costs Formulation

Hence, fixed transaction costs are formulated as:

$$\sum_{i=1}^{N} fc_{t,i}|a_{t,i} - A_{t,i}| = TC_t \tag{6}$$

Linear Transaction Costs

Linear transaction costs are formulated as:

$$\sum_{i=1}^{N} lc_{t,i} P_{t,i} t_{t,i} = TC_t \tag{7}$$

Table 2: Example to illustrate linear TC formulation

position index	current market price	trade	transaction amount	linear tc rate	TC amount
AAPL	140.880005	100	14088.0005	0.001	14.0880005
BA	77.599998	56	4345.599888	0.001	4.345599888
GM	176.100006	0	0	0.001	0
IBM	91.510002	-96	8784.960192	0.001	8.784960192
DD	33.990002	59	2005.410118	0.001	2.005410118
GOOG	106.279999	-20	2125.59998	0.001	2.12559998

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Notes for Efficient Frontiers

- 30 risk return scatter points for 30 DJIA component stocks
- two efficient frontiers
- 10 rainbow colored triangles

  "C": indifferent to the transaction cost, no Δ constraint
  "G": even more generous, no K constraint
  "Δ": position change constraint
- Difference between "Buy and Hold" and "$\Delta = 0$" strategy
- Corresponding black asterisks for net in transaction cost portfolio performances

Recall. . .

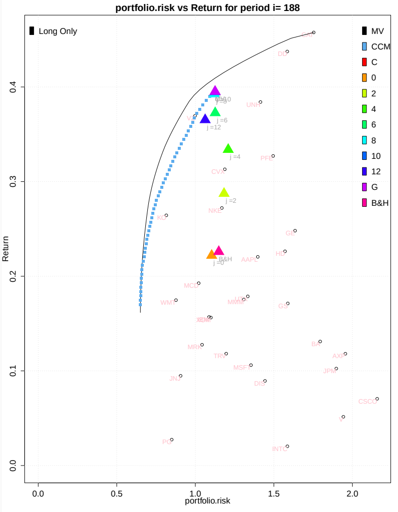
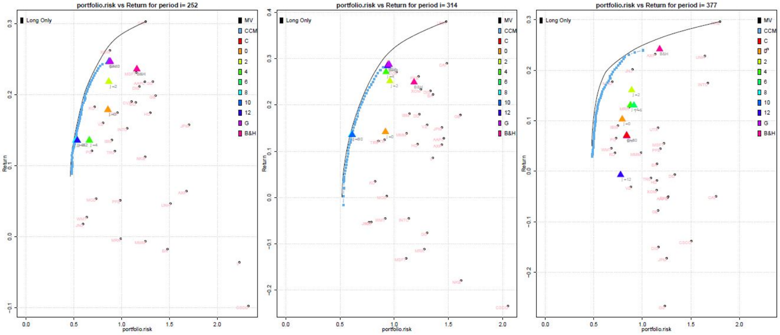


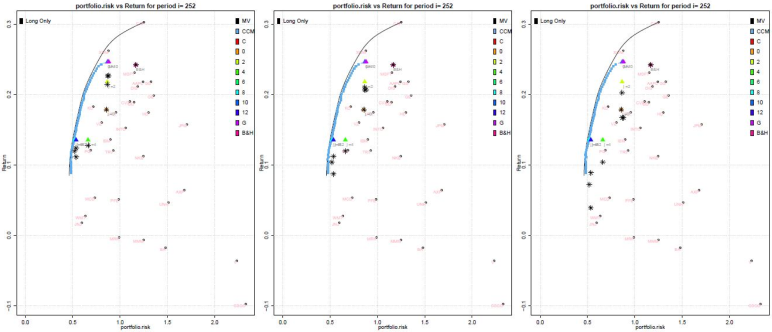
Figure 4: Efficient Frontier for different at period i = 188



(a) i = 252 (b) i = 314 (c) i = 377

Fig. 4.3 Efficient Frontier comparison for different time periods i = 252, 314, 377 with no transaction costs

Efficient Frontiers for various rates



(a) \$2.5 per position change (b) \$5 per position change (c) \$10 per position change

Fig. 4.4 Efficient Frontier for different strategies at period i = 252 with fixed transaction costs

Observations for Efficient Frontiers Comparison

- The performance of strategy vary from time to time, depends largely on data input.
  In other words, there is no best Δ.
- The effect of transaction costs on the portfolio performance depends largely on Δ and is linear with the fixed hurdle fee.
  e.g. no drop of "Buy and Hold" and "$\Delta = 0$"

Data Sets

dynamic

- Data Set
- Time window (Financial Crisis)
- General settings
  - $W_0 = V_0 = \$1000,000$
  - $K = 6$
  - 2.5% lower bounds and 20% upper bounds on weight for all of the positions.

Strategies...

Designing philosophy:

1. MV model in-cooperating with transaction costs.
2. MV model.
3. Cardinality constrained MV model.
4. CCMV with transaction costs.
5. MV with no weight bounds constraints.
6. Replications with net in transaction costs effects

Observations for Dynamic Performance Comparison

- a powerful evidence against the frictionless assumption in MV model. Portfolios in perfect and frictionless world, always performs better than in real world
- Portfolios show similar evolution trends (all under the mean variance framework)
  e.g. all suffer a deep loss during the financial crisis

Reasons for weight bounds

Now focus on *opMV* and *opMV.noPos*, observing that:

- purple is always above the orange line over the entire 10 years

  implies that weight constraints do limit a lot of the portfolio performance
- But, we have to impose the weight bound constraint. Why?

  dynamic1, *dynamic2*

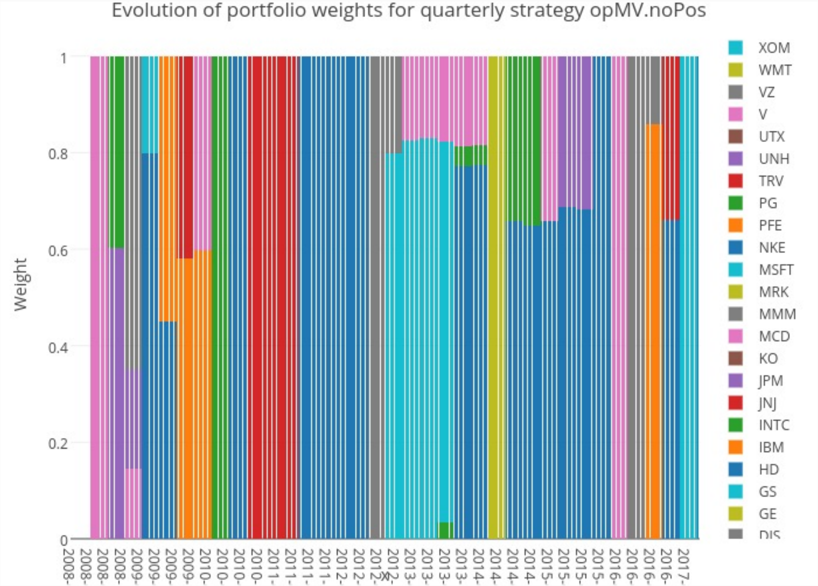


Figure 5: opMV.noPos Weight Evolution

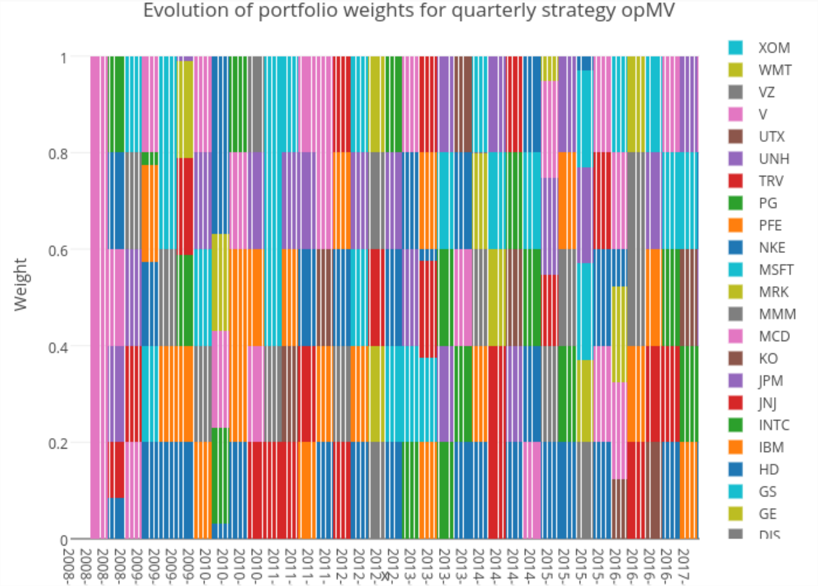


Figure 6: opMV Weight Evolution

Astonishing about Cardinality strategy

Now focus on *CCMV*

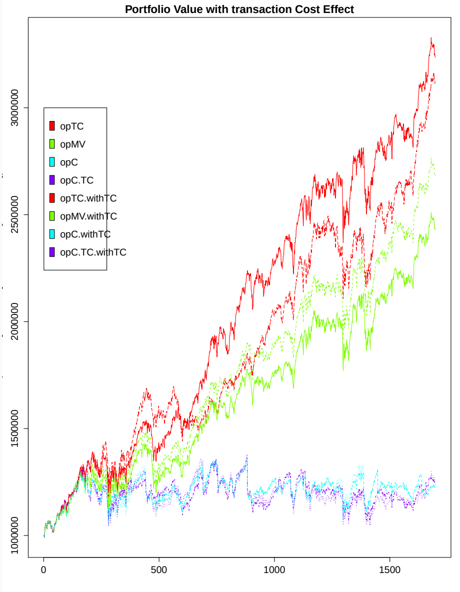
- one could hardly note the differences between green opC and orange opMV
- one could easily distinguish gray opC.withTC and pink opMV.withTC

Possible explanations:

- Cardinality Constraint on open positions further limits the possible TC incurred

- Is this true ?

Closer to the effect of transaction costs. . .



Astonishing about Cardinality strategy (continue)

This finding is exciting and evolutionary because theoretically, MV would always outperforms CCMV model, however here after incorporating the transaction costs, there is a complete reversal of the story.

Side Constraints

Examples

- clustering algorithm
- transaction cost
- turnover constraints
- limit the number of trades

Benefits

- release computational burden **feasibility**
- the same desired outcome: i.e. small portfolio size
- improve efficiency and speed "time is money"

Potential Problems

Unstable portfolio size (Results to be shown later)

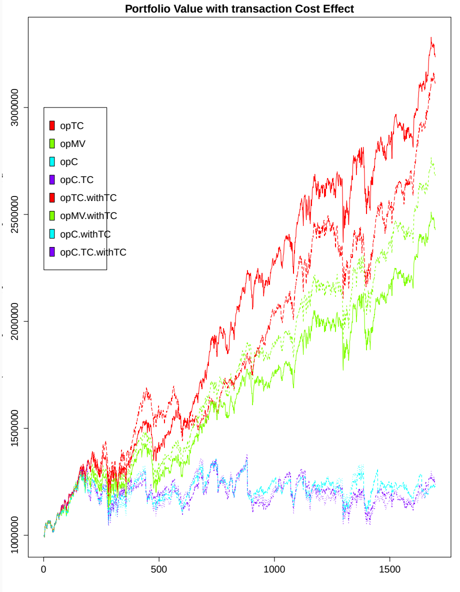


Figure 8: Focused Monetary portfolio value over one single investment period

Investigation on side constraints:

- Once cardinality constraints are already present in the model, adding transaction costs constraints does not change the result a lot
- Only imposing transaction costs constraints does not guarantee a stable desired level of portfolio size
- However, imposing only cardinality constraints
  - not only restricts the portfolio size to be exactly the number of positions the investor would like to hold
  - but it simultaneously prevents extreme turnovers or large amount of transactions.

Hence, cardinality in this sense is a much **better** constraint than others.

Now, let's conduct some sensitivity analyses...

Sensitivity Analysis:

Comparison Group 1: rebalancing frequency

- Monthly;
- Quarterly;

Figures :

dynamic monthly, *dynamic Quarterly*.

Sensitivity Analysis:

Comparison Group 1: rebalancing frequency

Observations :

e.g. Compare *opMV.noPos* and *opMV.noPos.withTC* in month and quarters.

Findings :

- For : Frequently checking the portfolio prevents extreme cases.
- Against : Frequently rebalancing incurs larger TC.

Sensitivity Analysis:

Comparison Group 2: Transaction Costs **TYPE**

- Fixed cost rate: \$100 per position change;
- Linear cost rate: \$10 basis point of the transaction amount.

Figures :
quarterly fixed 100, *quarterly linear*

Sensitivity Analysis:

Comparison Group 2: Transaction Costs **TYPE**

Findings :

- Not too much reference standards because of their very different structure
- One-to-one correspondence are more obvious for the linear type of transaction costs due to the existing linearity.

Sensitivity Analysis:

Comparison Group 3: Transaction Costs **MAGNITUDE**

- Fixed cost rate: $5 per position change;
- Fixed cost rate: $100 per position change;

Figures :
dynamic 5, *dynamic 100*.

Sensitivity Analysis:

Comparison Group 3: Transaction Costs **MAGNITUDE**

Findings :

- the rate of $5 per position being relatively subtle compared with the large portfolio capital in million dollar scale
- the drops of "with.TC" are also small compared with that of the $ 100 magnitude.

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interesting findings on Portfolio size ...

Table 8
Effect of Diversification

Number of Securities	Expected Portfolio Variance	Variance in Variance	Total Risk
1..........	46.619	1,411.041	46.811
2..........	26.839	201.963	26.934
4..........	16.948	31.553	16.996
6..........	13.651	11.184	13.683
8..........	12.003	5.477	12.027
10..........	11.014	3.186	11.033
20..........	9.036	.623	9.045
50..........	7.849	.075	7.853
100..........	7.453	.013	7.455
200..........	7.255	.001	7.256
500..........	7.137	.000	7.137
1,000..........	7.097	.000	7.097
Minimum......	7.070	.000	7.070

NOTE.—Parameters based on 3,290 securities values shown in table 5.

Figure 9: Effect of Diversification, Elton and Gruber [1]

interesting findings on Portfolio size . . .

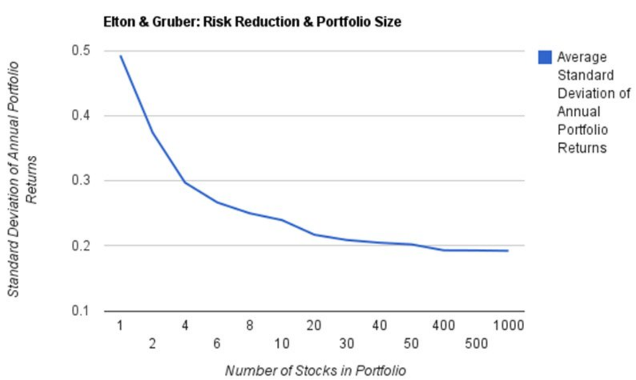

Figure 10: Risk reduction and portfolio size

. . . so Cardinality Problem is important and meaningful !

- the very DEEP reason: existence of Transaction costs
- Findings on Portfolio size: diversification effects are limited to a certain degree
- Deficiencies of other side constraints

How to choose an appropriate K ?

- Science: Look for market clues (e.g. clustering analyses. . .)
- Art: Investor's risk attitudes

Further Applications. . .

Figure 11: Elite team

how to choose these 7 students ?
how to maintain these 7 students ?

Further Applications. . .

Practical objectives on the implementation of such an *elite stream*:

1. Stability

   ensures a degree of sense of security for students currently within the group, which in turn is how it attracts those currently out of the group;
2. Competitiveness

   facilitate competitiveness and thus increasing the average performance of the whole class

More about position change...

How to choose an appropriate Δ ?

- Different levels of Δ appeal to different investors
- Δ closer to 0: an **individual** investor with relatively small amount of capital to choose a Δ value closer to zero since the unnecessary yet large amount transaction fees would eat up a lot, if not all, of his or her capital .
- Larger Δ: institutional investors or individuals with large amount of capital available may be indifferent of TC

Model evaluation

Advances

- Guarantee a solution with stable portfolio size
- Solve the problem of "how to maintain":
  Which out, which in
- Highly custimized and flexible
- even more applications

Further developments

- Needs to stick to the strategy for the whole investment horizon
- could be more more logical, flexible and profitable if one is able to modify the strategy according to the information innovation
- Yet, difficult:: Δ no longer a constant but a random variable

Thank you and enjoy reading!

Questions?

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